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A-LEVEL MATHEMATICS • FURTHER PURE 3

# Hyperbolic Functions

Worksheet

Pearson Edexcel International A Level (WFM03) • Calculator allowed: show every step

Free Hyperbolic Functions worksheet for A-Level and International A-Level Maths, with full worked solutions


A **Examples** & Your turn page 2

B **Fluency** core skills page 4

C **Reasoning** explain it page 6

D **Problem** solving page 7

E **Exam** practice page 9


Tap a section to jump straight to it

**By the end of this worksheet you can:** use the exponential definitions • write the inverse functions as logarithms • prove identities • *Stretch:* solve equations exactly, using identities and the forms $R\cosh(x \pm \alpha)$ and $R\sinh(x \pm \alpha)$

## Which questions are for me?

CORE The methods every A-level student must secure. Start here.

STRETCH Definitions, inverse functions in logarithmic form, identities, and equations with every invalid root rejected.

Every section has both, clearly labelled. Questions get harder as you go, and every skill in Sections A to D is needed in Section E.

## Track your progress

| Section | B | C | D | E |
|---|---|---|---|---|
| Core marks | 23 | 10 | 13 | 13 |
| Stretch marks | 16 | 14 | 17 | 9 |
| My score | | | | |


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# A Worked examples & Your turn

Read each example carefully. Cover it up, then answer the question beside it using the same method.

**CORE** *Secure the method*

### Worked example A1: exact values

Find $\sinh(\ln 2)$ and $\cosh(\ln 2)$

$e^{\ln 2} = 2,\ e^{-\ln 2} = \frac{1}{2}$

$\frac{1}{2}\left(2 - \frac{1}{2}\right),\ \frac{1}{2}\left(2 + \frac{1}{2}\right)$

$\frac{3}{4},\ \frac{5}{4}$

### Your turn A1

Find $\cosh(\ln 4)$ and $\tanh(\ln 4)$

### Worked example A2: an identity from the definitions

Prove that $\sinh 2x \equiv 2\sinh x \cosh x$

$2 \times \frac{e^x - e^{-x}}{2} \times \frac{e^x + e^{-x}}{2} = \frac{e^{2x} - e^{-2x}}{2}$

$\sinh 2x$

### Your turn A2

Prove that $\cosh 2x \equiv \cosh^2 x + \sinh^2 x$

### Worked example A3: a logarithmic form

Find $\operatorname{arsinh} \frac{3}{4}$ in its simplest form.

$\ln\left(\frac{3}{4} + \sqrt{\frac{9}{16} + 1}\right) = \ln\left(\frac{3}{4} + \frac{5}{4}\right)$

$\ln 2$

### Your turn A3

Find $\operatorname{artanh} \frac{3}{5}$ in its simplest form.

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## STRETCH

### Worked example A4: a quadratic in $e^x$

Solve $5\sinh x - \cosh x = 5$

$2e^x - 3e^{-x} = 5$, *so* $2u^2 - 5u - 3 = 0$ *with* $u = e^x$

$u = 3$ (reject $u = -\frac{1}{2}$)

$x = \ln 3$

### Your turn A4

Solve $3\cosh x - \sinh x = 3$

### Worked example A5: a quadratic in $\cosh x$

Solve $2\cosh^2 x - 7\cosh x + 3 = 0$

$\cosh x = 3$ *or* $\frac{1}{2}$; *reject* $\frac{1}{2}$ *since* $\cosh x \geqslant 1$

$x = \pm\ln(3 + 2\sqrt{2})$

### Your turn A5

Solve $\sinh^2 x - 2\sinh x - 3 = 0$

### Worked example A6: tanh and sech

Solve $\tanh x = \frac{1}{3}$

$\frac{1}{2}\ln\frac{1+\frac{1}{3}}{1-\frac{1}{3}}$

$x = \frac{1}{2}\ln 2$

### Your turn A6

Solve $\operatorname{sech} x = \frac{3}{5}$

### Worked example A7: the form $R\cosh(x+\alpha)$

Write $10\cosh x + 6\sinh x$ as $R\cosh(x+\alpha)$

$R\cosh\alpha = 10$, $R\sinh\alpha = 6$: $R^2 = 100 - 36$, $\tanh\alpha = \frac{3}{5}$

$8\cosh(x + \ln 2)$

### Your turn A7

Write $13\cosh x + 5\sinh x$ as $R\cosh(x+\alpha)$

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# B Fluency

Each question is a designed sequence. From one part to the next, ask yourself: what changed, and what did it do to the answer?

**CORE** *Secure the method*

**B1** Find the exact value of

**(a)** $\sinh(\ln 5)$

**(b)** $\cosh(\ln 5)$

**(c)** $\tanh(\ln 5)$

**(d)** $\operatorname{sech}(-\ln 5)$

**[4 marks]**

**B2** Write in logarithmic form

**(a)** $\operatorname{arsinh} 1$

**(b)** $\operatorname{arcosh} 3$

**(c)** $\operatorname{artanh} \frac{1}{3}$

**(d)** $\operatorname{arsinh}(-2)$

**[4 marks]**

**B3** Use Osborn's rule to write down the hyperbolic identity for

**(a)** $\cos 2x \equiv 2\cos^2 x - 1$

**(b)** $\sin 3x \equiv 3\sin x - 4\sin^3 x$

**(c)** $\cos 3x \equiv 4\cos^3 x - 3\cos x$

**(d)** $\sin^2 x \equiv \frac{1}{2}(1 - \cos 2x)$

**[4 marks]**

**B4** Solve, giving exact answers

**(a)** $\sinh x = 2$

**(b)** $\cosh x = 2$

**(c)** $4\sinh x + 2\cosh x = 3$

**(d)** $\cosh x - \sinh x = 2$

**[4 marks]**

**B5**

**(a)** State the range of $2 + \sinh x$

**(b)** State the range of $4 - 3\tanh x$

**(c)** Explain why $\cosh x = \sinh x$ has no solutions.

**[3 marks]**

**B6** Solve, giving exact answers

**(a)** $\cosh^2 x = 4$

**(b)** $2\sinh^2 x + 5\sinh x = 3$

**(c)** $\cosh 2x = 17$

**(d)** $\tanh^2 x = \frac{1}{4}$

**[4 marks]**

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## STRETCH

**B7**

**(a)** Show that $\cosh x + \sinh x = e^x$

**(b)** Show that $\cosh x - \sinh x = e^{-x}$

**(c)** Hence show that $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$

**(d)** Solve $\cosh 3x + \sinh 3x = 8$

**[4 marks]**

**B8**

**(a)** Prove that $\coth^2 x - \operatorname{cosech}^2 x \equiv 1$

**(b)** Write $\coth^2 x - 3\operatorname{cosech} x = 5$ as a quadratic in $\operatorname{cosech} x$

**(c)** Solve this quadratic.

**(d)** Find the exact values of $x$

**[4 marks]**

**B9** $17\cosh x + 8\sinh x \equiv R\cosh(x+\alpha)$, $R > 0$

**(a)** Find $R$

**(b)** Find $\alpha$ as a logarithm.

**(c)** State the least value of $17\cosh x + 8\sinh x$

**(d)** Solve $17\cosh x + 8\sinh x = 17$

**[4 marks]**

**B10**

**(a)** State the domain of $\operatorname{arcosh} x$

**(b)** State the domain of $\operatorname{artanh} x$

**(c)** Use the logarithmic form to find $\operatorname{artanh} \frac{1}{2}$

**(d)** Solve $\operatorname{artanh} x = \ln 2$

**[4 marks]**

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# C Reasoning

Show your working and explain your thinking. A correct answer without a reason does not score full marks.

## CORE *Secure the method*

**C1** Asked to prove $\sinh 2x \equiv 2\sinh x \cosh x$ *from the definitions*, Amir quotes Osborn's rule. Explain why this does not answer the question. **[2 marks]**

**C2** Bea solves $\cosh x = -2$ and writes $x = \ln\left(-2+\sqrt{3}\right)$. Explain her error. **[2 marks]**

**C3** Cal writes $\cosh x = e^x + e^{-x}$ and solves $\cosh x = \frac{5}{2}$, obtaining $e^x = 2$ or $\frac{1}{2}$

Explain his error, and solve $\cosh x = \frac{5}{2}$ correctly. **[3 marks]**

**C4** Solving $e^{2x} + 2e^x - 15 = 0$, Dev writes $x = \ln 3$ or $x = \ln(-5)$

Explain his error, and give the solution. **[3 marks]**

## STRETCH

**C5** Eve gives the only solution of $\cosh x = 5$ as $x = \ln\left(5+2\sqrt{6}\right)$

Explain her error, and give all the solutions. **[3 marks]**

**C6** From $\cosh 2x = 7$, Fay obtains $\cosh^2 x = 4$ and writes $x = \operatorname{arcosh} 4$

Explain her error, and solve the equation. **[3 marks]**

**C7** Gus solves $\tanh x = 2$ and writes $x = \frac{1}{2}\ln(-3)$

Explain why there is no solution. **[3 marks]**

**C8** **(a)** Let $y = \operatorname{artanh} x$, so that $x = \tanh y$. Using the exponential definition, prove that $\operatorname{artanh} x = \frac{1}{2}\ln\frac{1+x}{1-x}$

**(3)**

**(b)** Explain why this requires $-1 < x < 1$

**(2)**

**[5 marks]**

# D Problem solving

You decide the method. Read the whole question first, and show every step.

## CORE *Secure the method*

**D1** Determine the exact solutions of the equation

$$2\sinh^2 x = 5\cosh x - 5$$

giving any non-zero answers as natural logarithms in simplest form. **[3 marks]**

**D2** **(a)** Use the definitions of $\sinh x$ and $\cosh x$ in terms of exponentials to show that

$$\cosh(A - B) \equiv \cosh A \cosh B - \sinh A \sinh B$$

**(2)**

**(b)** Hence determine the exact value of $x$ for which $\cosh(x - \ln 3) = 3\sinh x$, giving your answer as a single natural logarithm.

**(2)**

**[4 marks]**

**D3** *In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.*

Solve the equation

$$6\cosh x - 2\sinh x = 3e^x - 1$$

giving your answer as a natural logarithm in simplest form. **[3 marks]**

**D4** By writing the equation in terms of exponentials, show that the equation

$$3\cosh x - 2\sinh x = 2$$

has no real solutions. **[3 marks]**

## STRETCH

**D5** **(a)** Express $13\sinh x + 5\cosh x$ in the form $R\sinh(x + \alpha)$, where $R > 0$, giving $\alpha$ as a natural logarithm.

**(2)**

**(b)** Hence solve the equation $13\sinh x + 5\cosh x = 12$, giving your answer as a single natural logarithm.

**(2)**

**[4 marks]**

**D6** **(a)** Use the definition of $\cosh x$ in terms of exponentials to show that

$$4\cosh^3 x \equiv \cosh 3x + 3\cosh x$$

**(2)**

**(b)** Hence solve the equation $\cosh 3x = 9\cosh x$, giving your answers as natural logarithms in simplest form.

**(2)**

**[4 marks]**

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**D7** By first writing $\tanh x$ and $\operatorname{sech} x$ in terms of $e^x$, solve the equation

$$\tanh x + 3\operatorname{sech} x = 3$$

giving any non-zero answer as a natural logarithm. **[4 marks]**

**D8** **(a)** Use the definitions of hyperbolic functions in terms of exponentials to prove that

$$\cosh 2x \equiv 1 + 2\sinh^2 x$$

**(2)**

**(b)** Hence solve the equation $\cosh 2x = 7\sinh x - 2$, giving your answers as natural logarithms in simplest form.

**(3)**

**[5 marks]**

# E Exam practice

Exam-style questions in the style of WFM03. Write your answers in the spaces, and show every step.

CORE *Exam questions for everyone*

**E1** *In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.*

Solve the equation

$$3\cosh^2 x - 4\sinh x = 7$$

Give your answers as simplified natural logarithms.

**(Total for Question E1 is 6 marks)**

**E2** **(a)** Use the definitions of hyperbolic functions in terms of exponentials to show that

$$\sinh(A-B) \equiv \sinh A \cosh B - \cosh A \sinh B$$

**(2)**

**(b)** Hence express $5\sinh x - 3\cosh x$ in the form $R\sinh(x-\alpha)$, where $R > 0$, giving $\alpha$ in the form $\ln p$ where $p$ is an integer.

**(3)**

**(c)** Hence solve the equation $5\sinh x - 3\cosh x = 4$, giving your answer in the form $\ln\left(a + b\sqrt{2}\right)$, where $a$ and $b$ are integers.

**(2)**

**(Total for Question E2 is 7 marks)**

STRETCH

**E3** **(a)** Use the definition of $\sinh x$ in terms of exponentials to prove that

$$4\sinh^3 x \equiv \sinh 3x - 3\sinh x$$

**(3)**

**(b)** Hence determine the exact solutions of the equation

$$\sinh 3x = 19\sinh x$$

giving your answers in terms of natural logarithms where appropriate.

**(3)**

**(c)** For the positive solution $x = \beta$ found in part (b), determine the exact value of $\cosh 3\beta$

**(3)**

**(Total for Question E3 is 9 marks)**

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