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A-LEVEL MATHEMATICS • FURTHER PURE 3

Further Matrix Algebra

Worksheet

Pearson Edexcel International A Level (WFM03) • Calculator allowed: show every step

Free Further Matrix Algebra worksheet for A-Level and International A-Level Maths, with full worked solutions

A — **Examples** & Your turn — page 2

B — **Fluency** core skills — page 4

C — **Reasoning** explain it — page 7

D — **Problem** solving — page 8

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Tap a section to jump straight to it

By the end of this worksheet you can: find 3×3 determinants and inverses • find eigenvalues and eigenvectors • *Stretch:* use given eigenvectors, transform lines and planes, and diagonalise symmetric matrices.

Which questions are for me?

CORE The methods every A-level student must secure. Start here.

STRETCH Determinants, inverses, transformations of space, eigenvalues and eigenvectors, and diagonalising symmetric matrices.

Every section has both, clearly labelled. Questions get harder as you go, and every skill in Sections A to D is needed in Section E.

Track your progress

Section	B	C	D	E
Core marks	23	10	13	13
Stretch marks	16	14	17	9
My score				

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Name ..... Class ..... Date .....

A Worked examples & Your turn

Read each example carefully. Cover it up, then answer the question beside it using the same method.

CORE *Secure the method*

Worked example A1: a determinant

Find $\det\begin{pmatrix}2&1&3\\0&4&1\\1&2&1\end{pmatrix}$

$2(4-2)-1(0-1)+3(0-4)$

-7

Your turn A1

Find $\det\begin{pmatrix}1&0&2\\3&1&1\\2&4&0\end{pmatrix}$

Worked example A2: a singular matrix

Find k so that $\begin{pmatrix}k&1&2\\1&2&1\\3&1&1\end{pmatrix}$ is singular.

$\det = k(1)-1(1-3)+2(1-6) = k-8$

$k = 8$

Your turn A2

Find k so that $\begin{pmatrix}1&k&0\\2&1&3\\1&1&2\end{pmatrix}$ is singular.

Worked example A3: an inverse

Find the inverse of $\begin{pmatrix}2&0&0\\0&1&3\\0&0&1\end{pmatrix}$

Undo each part: halve x; $y+3z$ needs $-3z$

$\begin{pmatrix}\frac{1}{2}&0&0\\0&1&-3\\0&0&1\end{pmatrix}$

Your turn A3

Find the inverse of $\begin{pmatrix}1&2&0\\0&1&0\\0&0&4\end{pmatrix}$

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STRETCH

Worked example A4: checking an eigenvector

Show that $\begin{pmatrix}1\\0\\1\end{pmatrix}$ is an eigenvector of $\begin{pmatrix}2&0&1\\0&3&0\\1&0&2\end{pmatrix}$

$$\begin{pmatrix}2&0&1\\0&3&0\\1&0&2\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}3\\0\\3\end{pmatrix}$$

$\boxed{\lambda = 3}$

Your turn A4

Show that $\begin{pmatrix}1\\1\\1\end{pmatrix}$ is an eigenvector of $\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}$

Worked example A5: eigenvalues

Find the eigenvalues of $\begin{pmatrix}2&0&0\\1&3&0\\4&5&1\end{pmatrix}$

Triangular: $\det(\mathbf{M}-\lambda\mathbf{I}) = (2-\lambda)(3-\lambda)(1-\lambda)$

$\boxed{1,\ 2,\ 3}$

Your turn A5

Find the eigenvalues of $\begin{pmatrix}1&4&2\\0&-2&7\\0&0&5\end{pmatrix}$

Worked example A6: an eigenvector

Find an eigenvector of $\begin{pmatrix}3&0&0\\0&1&2\\0&2&1\end{pmatrix}$ for $\lambda = -1$

$(\mathbf{M}+\mathbf{I})\mathbf{x} = \mathbf{0}$: $4x = 0,\ 2y+2z = 0$

$\boxed{\begin{pmatrix}0\\1\\-1\end{pmatrix}}$

Your turn A6

Find an eigenvector of $\begin{pmatrix}1&0&2\\0&3&0\\2&0&1\end{pmatrix}$ for $\lambda = -1$

Worked example A7: normalising

Normalise $\begin{pmatrix}1\\2\\2\end{pmatrix}$

Length $\sqrt{1+4+4} = 3$*; divide by it*

$\boxed{\frac{1}{3}\begin{pmatrix}1\\2\\2\end{pmatrix}}$

Your turn A7

Normalise $\begin{pmatrix}1\\-1\\2\end{pmatrix}$

B Fluency

Each question is a designed sequence. From one part to the next, ask yourself: what changed, and what did it do to the answer?

CORE *Secure the method*

B1 Find the determinant of

(a) $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{pmatrix}$

(b) $\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}$

(d) $2\mathbf{A}$, using (b)

[4 marks]

B2 $\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

(a) Find $\det \mathbf{A}$

(b) Find the matrix of cofactors.

(c) Find the adjugate.

(d) Find $\mathbf{A}^{-1}$

[4 marks]

B3 $\det \mathbf{A} = 3$ and $\det \mathbf{B} = -2$ for 3×3 matrices $\mathbf{A}$ and $\mathbf{B}$. Find

(a) $\det(\mathbf{AB})$

(b) $\det\left(\mathbf{A}^{-1}\right)$

(c) $\det\left(\mathbf{A}^{\mathrm{T}}\mathbf{B}\right)$

(d) $\det(2\mathbf{A})$

[4 marks]

B4 Find the values of k for which each matrix is singular, or show there are none.

(a) $\begin{pmatrix} k & 2 & 1 \\ 0 & k & 3 \\ 1 & 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 1 & k \\ 2 & k & 1 \\ 1 & 1 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} k & 0 & 1 \\ 0 & k & 1 \\ 1 & 1 & k \end{pmatrix}$

(d) State what singular means for a transformation.

[4 marks]

B5 $\mathbf{M} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$. Find the eigenvalue for each eigenvector.

(a) $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$, and explain what (b) and (c) show. **[3 marks]**

B6 $\mathbf{M} = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 4 \end{pmatrix}$

(a) Write down $\det(\mathbf{M} - \lambda\mathbf{I})$

(b) Factorise it.

(c) State the eigenvalues.

(d) Find an eigenvector for the largest eigenvalue.

[4 marks]

STRETCH

B7 $\mathbf{M} = \begin{pmatrix} 1 & a & 2 \\ 2 & b & 1 \\ 1 & 1 & 3 \end{pmatrix}$ has eigenvector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(a) Find the eigenvalue.

(b) Find a and b

(c) Find $\det\mathbf{M}$

(d) Use $\det\mathbf{M}$ and the trace to find the other eigenvalues.

[4 marks]

B8 $\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ from B2

(a) Write $x + 2z = 5$, $2x + y = 4$, $y + z = 3$ as $\mathbf{Ax} = \mathbf{c}$

(b) Solve for $\mathbf{x}$ using $\mathbf{A}^{-1}$

(c) Check your answer.

(d) Interpret the answer geometrically.

[4 marks]

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B9 The transformation T has matrix $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

(a) Find the image of $(1, 2, 3)$

(b) Find the image of the line $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

(c) Find the point with image $(3, 4, 1)$

(d) Find det and say what it means for volumes.

[4 marks]

B10 $\mathbf{P} = \frac{1}{3} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$

(a) Find $\mathbf{P}^{\mathrm{T}}\mathbf{P}$

(b) Hence write down $\mathbf{P}^{-1}$

(c) Find $\det \mathbf{P}$

(d) Explain why the columns of $\mathbf{P}$ are unit and mutually perpendicular.

[4 marks]

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C Reasoning

Show your working and explain your thinking. A correct answer without a reason does not score full marks.

CORE *Secure the method*

C1 To find the eigenvalues of $\mathbf{M}$, Amir solves $\det\mathbf{M} = 0$. Explain his error. **[2 marks]**

C2 Bea divides her matrix of cofactors by $\det\mathbf{A}$ and calls it $\mathbf{A}^{-1}$. Explain her error. **[2 marks]**

C3 Cal writes $(\mathbf{AB})^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}$

Explain his error, and show that the correct form works. **[3 marks]**

C4 Dev normalises $\begin{pmatrix}2\\1\\2\end{pmatrix}$ by multiplying it by its length, giving $\begin{pmatrix}6\\3\\6\end{pmatrix}$

Explain his error, and correct it. **[3 marks]**

STRETCH

C5 For $\mathbf{M} = \begin{pmatrix}1&0&2\\0&3&0\\2&0&1\end{pmatrix}$ and $\lambda = -1$, Eve solves $(\mathbf{M}+\mathbf{I})\mathbf{x} = -\mathbf{x}$

Explain her error, and find an eigenvector. **[3 marks]**

C6 Given $k^2 = 9$ and $k < 0$, Fay takes $k = 3$ because it is "simpler". Explain her error. **[3 marks]**

C7 For $\mathbf{M} = \begin{pmatrix}1&2&2\\2&1&2\\2&2&1\end{pmatrix}$, Gus chooses $\begin{pmatrix}1\\-1\\0\end{pmatrix}$ and $\begin{pmatrix}1\\0\\-1\end{pmatrix}$ for the repeated eigenvalue -1 in an orthogonal $\mathbf{P}$

Explain the problem, and choose suitable eigenvectors. **[3 marks]**

C8 **(a)** Show that if $\mathbf{Mx} = \lambda\mathbf{x}$ then $\mathbf{M}^2\mathbf{x} = \lambda^2\mathbf{x}$

(2)

(b) $\mathbf{M}$ is symmetric, with $\mathbf{Mx}_1 = \lambda_1\mathbf{x}_1$, $\mathbf{Mx}_2 = \lambda_2\mathbf{x}_2$ and $\lambda_1 \neq \lambda_2$. Show that $\mathbf{x}_1 \cdot \mathbf{x}_2 = 0$

(3)

[5 marks]

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D Problem solving

You decide the method. Read the whole question first, and show every step.

CORE *Secure the method*

D1 *In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.*

$\mathbf{A} = \begin{pmatrix} 1 & k & 2 \\ 0 & 2 & 1 \\ 3 & 1 & k \end{pmatrix}$, where k is a constant.

Determine, in terms of k, the determinant of $\mathbf{A}$, showing your expansion clearly, and hence find the value of k for which $\mathbf{A}$ is singular. **[3 marks]**

D2 $\mathbf{A} = \begin{pmatrix} x & 1 & 1 \\ -2 & x & 0 \\ 1 & 0 & 1 \end{pmatrix}$, where x is a real constant.

(a) By finding $\det \mathbf{A}$ in terms of x and completing the square, show that $\mathbf{A}$ is non-singular for all real values of x

(2)

(b) Determine $\mathbf{A}^{-1}$ when $x = 1$

(2)

[4 marks]

D3 *In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.*

The transformation represented by $\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$ maps the plane Π_1, with equation $x + y + z = 4$, to the plane Π_2

The point (x, y, z) on Π_1 maps to the point (u, v, w) on Π_2. By first expressing x, y and z in terms of u, v and w, determine a Cartesian equation for Π_2 in the form $au + bv + cw = d$, where a, b, c and d are integers.

[3 marks]

D4 $\mathbf{M} = \begin{pmatrix} 2 & k & 0 \\ k & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, where k is a positive constant.

Given that 7 is an eigenvalue of $\mathbf{M}$, and substituting $\lambda = 7$ before expanding the determinant,

(a) determine the exact value of k

(2)

(b) determine the other eigenvalues of $\mathbf{M}$

(1)

[3 marks]

STRETCH

D5 $\mathbf{M} = \begin{pmatrix} 1 & a & 3 \\ 2 & b & 0 \\ 1 & 1 & 1 \end{pmatrix}$, where a and b are constants.

Given that $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{M}$,

(a) determine the corresponding eigenvalue, and the values of a and b

(2)

(b) By forming the characteristic equation of $\mathbf{M}$ and using your answer to part (a), show that $\mathbf{M}$ has no other real eigenvalues.

(2)

[4 marks]

D6 $\mathbf{M} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ 1 & 2 & 1 \end{pmatrix}$

(a) Determine the eigenvalues of $\mathbf{M}$

(2)

(b) Determine an eigenvector for each distinct eigenvalue, and explain how many independent eigenvectors there are for the repeated eigenvalue.

(2)

[4 marks]

D7 $\mathbf{T} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 3 \end{pmatrix}$ and $\mathbf{U} = \frac{1}{7}\begin{pmatrix} 3 & a & 1 \\ 1 & 6 & -2 \\ b & 1 & c \end{pmatrix}$, where a, b and c are constants.

Given that $\mathbf{TU} = \mathbf{I}$, where $\mathbf{I}$ is the 3×3 identity matrix, determine the values of a, b and c by considering suitable entries of the product, and state what the matrix $\mathbf{U}$ represents. **[4 marks]**

D8 $\mathbf{M} = \begin{pmatrix} k & 1 & 0 \\ 1 & k & 0 \\ 0 & 0 & 4 \end{pmatrix}$, where k is a constant.

(a) Show that the eigenvalues of $\mathbf{M}$ are 4, $k+1$ and $k-1$

(2)

(b) Given that $\mathbf{M}$ has a repeated eigenvalue, determine the possible values of k and, for each, all the eigenvalues of $\mathbf{M}$

(3)

[5 marks]

E Exam practice

Exam-style questions in the style of WFM03. Write your answers in the spaces, and show every step.

CORE *Exam questions for everyone*

E1 In this question you must show all stages of your working.

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & k \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{pmatrix}, \text{ where } k \text{ is a constant.}$$

(a) Determine the value of k for which the matrix $\mathbf{A}$ is singular.

(2)

Given that $\mathbf{A}$ is non-singular,

(b) determine $\mathbf{A}^{-1}$, giving your answer in simplest form in terms of k

(4)

(Total for Question E1 is 6 marks)

E2 The symmetric matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$$

(a) By expanding $\det(\mathbf{M} - \lambda\mathbf{I})$, show that the eigenvalues of $\mathbf{M}$ satisfy $(\lambda - 1)(\lambda - 3)(\lambda - 4) = 0$

(4)

(b) Determine a normalised eigenvector of $\mathbf{M}$ corresponding to the eigenvalue 1, and verify that it is perpendicular to the eigenvector $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ corresponding to the eigenvalue 4

(3)

(Total for Question E2 is 7 marks)

STRETCH

E3 The symmetric matrix $\mathbf{M}$ represents a linear transformation of three-dimensional space and is given by

$$\mathbf{M} = \begin{pmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{pmatrix}$$

(a) Show that $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ is an eigenvector of $\mathbf{M}$, stating the corresponding eigenvalue.

(1)

(b) Determine the other two eigenvalues of $\mathbf{M}$, showing clearly how you use the characteristic equation

(3)

(c) Determine a normalised eigenvector for each eigenvalue of $\mathbf{M}$

(3)

(d) Hence determine an orthogonal matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{\mathrm{T}}\mathbf{M}\mathbf{P} = \mathbf{D}$, making clear how the columns of $\mathbf{P}$ correspond to the entries of $\mathbf{D}$

(2)

(Total for Question E3 is 9 marks)

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