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A-LEVEL MATHEMATICS • FURTHER PURE 1

Proof

Worked Solutions

Pearson Edexcel International A Level (WFM01) • Calculator allowed: show every step

Full worked solutions for the free Proof worksheet, A-Level and International A-Level Maths

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Tap a section to jump straight to it

Full worked solutions for every question on the Further Pure 1 Proof worksheet, in the same order and with the same question numbers.

How to mark your work. Tick each step you got right, not just the final answer. At A-level most marks are for method, so a wrong final answer can still earn most of the marks. If you lost marks, find the *first* line where your working is different from ours. That is the thing to fix.

In Section E: **M** = method mark, **dM** = method mark that depends on the previous M mark, **A** = accuracy mark (needs the M mark before it), **B** = independent mark.

Question sheet | Exam questions by topic | Further Pure 1 exam questions | More worksheets

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A Your turn answers

Compare each line of your working with the solution.

CORE

A1 $n = 1$: LHS = 2, RHS = $\frac{1}{2}(4) = 2$. Assume $\sum_{r=1}^{k}(3r-1) = \frac{1}{2}k(3k+1)$; $\sum_{r=1}^{k+1} = \frac{1}{2}k(3k+1)+3k+2 = \frac{1}{2}(3k^2+7k+4) = \frac{1}{2}(k+1)(3k+4)$

True for $n = 1$; if true for $n = k$ then true for $n = k+1$; so true for all $n \in \mathbb{Z}^+$

A2 f(1) = 8. Assume $8 \mid \text{f}(k)$; $\text{f}(k+1) = 9 \times 9^k - 1 = 9\text{f}(k) + 8$

True for $n = 1$; if true for $n = k$ then true for $n = k+1$; so true for all $n \in \mathbb{Z}^+$

A3 $n = 1$: $\begin{pmatrix}1 & 0\\ 1 & 2\end{pmatrix}\cdot\begin{pmatrix}1 & 0\\ 2^k-1 & 2^k\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 2\end{pmatrix} = \begin{pmatrix}1 & 0\\ 2^k-1+2^k & 2^{k+1}\end{pmatrix} = \begin{pmatrix}1 & 0\\ 2^{k+1}-1 & 2^{k+1}\end{pmatrix}$

True for $n = 1$; if true for $n = k$ then true for $n = k+1$; so true for all $n \in \mathbb{Z}^+$

STRETCH

A4 $u_1 = 3+2 = 5$. Assume $u_k = 3^k+2$; $u_{k+1} = 3(3^k+2)-4 = 3^{k+1}+2$

True for $n = 1$; if true for $n = k$ then true for $n = k+1$; so true for all $n \in \mathbb{Z}^+$

A5 f(1) = 8 + 27 = 35. $\text{f}(k+1) = 2 \times 2^{k+2} + 9 \times 3^{2k+1} = 2\text{f}(k) + 7 \times 3^{2k+1}$

True for $n = 1$; if true for $n = k$ then true for $n = k+1$; so true for all $n \in \mathbb{Z}^+$

A6 $n = 1$: $1 = 2! - 1$. Assume $\sum_{r=1}^{k} r\,r! = (k+1)! - 1$; add $(k+1)(k+1)!$: $(k+1)!(1+k+1) - 1 = (k+2)! - 1$

True for $n = 1$; if true for $n = k$ then true for $n = k+1$; so true for all $n \in \mathbb{Z}^+$

A7 f(1) = 0. $\text{f}(k+1) = (k+1)^3 - (k+1) = \text{f}(k) + 3k(k+1)$; $k(k+1)$ is even, so $3k(k+1)$ is a multiple of 6

True for $n = 1$; if true for $n = k$ then true for $n = k+1$; so true for all $n \in \mathbb{Z}^+$

B Fluency solutions

One mark for each correct part. The notes in italics explain why each sequence was chosen.

CORE

B1 **(a)** LHS = 1, RHS = $\frac{1}{6}(1)(2)(3) = 1$ **(b)** $7 - 1 = 6$ **(c)** $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$, the matrix itself **(d)** $2^1 + 1 = 3 = u_1$

Show both sides, with working.

B2 **(a)** $\boxed{5\mathrm{f}(k) + 4}$ **(b)** $\boxed{9\mathrm{f}(k) + 8}$ **(c)** $\boxed{8\mathrm{f}(k) + 7}$ **(d)** $\boxed{11\mathrm{f}(k) + 60}$

The added constant shows the divisor.

B3 **(a)** $\frac{1}{2}(k+1)(k+2)$ **(b)** $\frac{1}{4}(k+1)^2\left(k^2+4(k+1)\right) = \frac{1}{4}(k+1)^2(k+2)^2$ **(c)** $\frac{1}{6}(k+1)(2k^2+7k+6)$, and $2k^2+7k+6 = (k+2)(2k+3)$ **(d)** $k^2 + 2k + 1$

Take out the common factor $(k+1)$ *first.*

B4 **(a)** $\boxed{\begin{pmatrix} 9 & 8 \\ 0 & 1 \end{pmatrix}}$ **(b)** $\boxed{\begin{pmatrix} 27 & 26 \\ 0 & 1 \end{pmatrix}}$ **(c)** $\boxed{\begin{pmatrix} 3^n & 3^n - 1 \\ 0 & 1 \end{pmatrix}}$ **(d)** $\begin{pmatrix} 3^k & 3^k - 1 \\ 0 & 1 \end{pmatrix}\mathbf{M} = \begin{pmatrix} 3^{k+1} & 2 \times 3^k + 3^k - 1 \\ 0 & 1 \end{pmatrix}$

True for $n = 1$; if true for $n = k$ then true for $n = k + 1$; so true for all $n \in \mathbb{Z}^+$

Conjecture from small powers, then prove.

B5 **(a)** $\boxed{\mathbf{5,\ 7,\ 11}}$ **(b)** $\boxed{u_n = 2^{n-1} + 3}$ **(c)** $u_1 = 4$; $u_{k+1} = 2(2^{k-1} + 3) - 3 = 2^k + 3$

True for $n = 1$; if true for $n = k$ then true for $n = k + 1$; so true for all $n \in \mathbb{Z}^+$

A recurrence gives terms; induction proves the pattern.

B6 **(a)** $\boxed{5}$ **(b)** $8 \times 8^k - 3 \times 3^k = 8(8^k - 3^k) + 5 \times 3^k$ **(c)** both terms are multiples of 5

True for $n = 1$; if true for $n = k$ then true for $n = k + 1$; so true for all $n \in \mathbb{Z}^+$ **(d)** $4015 = 5 \times 803$

Rearranging to $8\mathrm{f}(k)$ *leaves a multiple of 5*

STRETCH

B7 **(a)** $\frac{1}{1 \times 3} = \frac{1}{3}$ **(b)** $\frac{k(2k+3)+1}{(2k+1)(2k+3)} = \frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$ **(c)** True for $n = 1$; if true for $n = k$ then true for $n = k + 1$; so true for all $n \in \mathbb{Z}^+$ **(d)** $\boxed{\frac{50}{101}}$

Factorise the numerator to cancel $(2k+1)$

B8 **(a)** $\begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}$ **(b)** $\mathbf{M}^k\mathbf{M}$ **(c)** $\begin{pmatrix}1 & 1+2(2^k-1)\\ 0 & 2^{k+1}\end{pmatrix} = \begin{pmatrix}1 & 2^{k+1}-1\\ 0 & 2^{k+1}\end{pmatrix}$

True for $n = 1$; if true for $n = k$ then true for $n = k+1$; so true for all $n \in \mathbb{Z}^+$

(d) $\boxed{2^n}$

The proved formula gives the determinant at once.

B9 **(a)** $f(1) = 9$ **(b)** $7^{k+1} + 3k + 2 - 7(7^k + 3k - 1) = 9 - 18k$ **(c)** $f(k+1) = 7f(k) + 9(1-2k)$

True for $n = 1$; if true for $n = k$ then true for $n = k+1$; so true for all $n \in \mathbb{Z}^+$

(d) $343 + 9 - 1 = 351 = 9 \times 39$

The linear term leaves a multiple of 9

B10 **(a)** $25 - 6$: $\boxed{19}$ **(b)** $3 - 2 = 1$, $9 - 4 = 5$ **(c)** assume true for $n = k$ and $k+1$: $u_{k+2} = 5(3^{k+1} - 2^{k+1}) - 6(3^k - 2^k) = 3^{k+2} - 2^{k+2}$

true for $n = 1, 2$, so for all $n \in \mathbb{Z}^+$

(d) each term uses the two before it, so the step needs two true cases

A second-order recurrence needs two basis cases.

C Reasoning solutions

Your wording can be different. Check that your reason makes the same point.

CORE

C1 The basis must be shown: $\boxed{n = 1\text{: LHS} = 1^3 = 1\text{, RHS} = \frac{1}{4}(1)(4) = 1\text{, so true for } n = 1}$

C2 The conclusion must link the steps:

$\boxed{\text{True for } n = 1\text{; if true for } n = k \text{ then true for } n = k+1\text{; so true for all } n \in \mathbb{Z}^+}$

C3 That assumes what is to be proved. Assume true for $n = k$, then show it is true for $n = k+1$

C4 The difference alone is not $f(k+1)$: $f(k+1) = f(k) + 5 \times 6^k$, a sum of multiples of 5

STRETCH

C5 Checking cases is not a proof. $n = 40$: $1600 + 40 + 41 = 41^2$: $\boxed{\text{not prime}}$

C6 Powers multiply, they do not add: $\mathbf{M}^{k+1} = \mathbf{M}^k\mathbf{M}$

C7 Divisibility by 6 needs a factor of 2 as well: $k(k+1)$ is a product of consecutive integers, so it is even, and $3k(k+1)$ is a multiple of 6

C8 $n = 1$: LHS = 4, RHS = 2(1)(2) = 4. Assume $\sum_{r=1}^{k}(6r^2 - 2r) = 2k^2(k+1)$

$$\sum_{r=1}^{k+1} = 2k^2(k+1) + 6(k+1)^2 - 2(k+1) = 2(k+1)(k^2+3k+2) = 2(k+1)^2(k+2)$$

True for $n = 1$; if true for $n = k$ then true for $n = k+1$; so true for all $n \in \mathbb{Z}^+$

D Problem solving solutions

There is often more than one correct method. Any complete, correct method scores full marks.

CORE

D1 f(1) = 9. Assume $3 \mid \mathrm{f}(k)$; $\mathrm{f}(k+1) = 4 \times 2^{2k+1} + 1 = 4\mathrm{f}(k) - 3$

True for $n = 1$; if true for $n = k$ then true for $n = k+1$; so true for all $n \in \mathbb{Z}^+$

D2 $n = 1$: $\begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 4^k & 4^k - 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4^{k+1} & 3 \times 4^k + 4^k - 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4^{k+1} & 4^{k+1} - 1 \\ 0 & 1 \end{pmatrix}$

True for $n = 1$; if true for $n = k$ then true for $n = k+1$; so true for all $n \in \mathbb{Z}^+$

D3 $n = 1$: LHS = 3, RHS = $\frac{3}{2}(2) = 3$. $\frac{3}{2}(3^k - 1) + 3^{k+1} = \frac{3}{2}(3^k - 1 + 2 \times 3^k) = \frac{3}{2}(3^{k+1} - 1)$

True for $n = 1$; if true for $n = k$ then true for $n = k+1$; so true for all $n \in \mathbb{Z}^+$

D4 $u_1 = 4 + 3 = 7$. $u_{k+1} = 4(4^k + 3) - 9 = 4^{k+1} + 3$

True for $n = 1$; if true for $n = k$ then true for $n = k+1$; so true for all $n \in \mathbb{Z}^+$

STRETCH

D5 f(1) = 0 is trivial, so start at $n = 2$: f(2) = 49 − 13 = 36. $\mathrm{f}(k+1) = 7 \times 7^k - 6k - 7 = 7\mathrm{f}(k) + 36k$

true for $n = 2$; if true for $n = k$ then true for $n = k+1$; so true for all $n \geqslant 2$

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D6 $n=1$: $\begin{pmatrix}-1 & 2\\ 0 & 1\end{pmatrix}.\begin{pmatrix}(-1)^k & 1-(-1)^k\\ 0 & 1\end{pmatrix}\mathbf{M}=\begin{pmatrix}(-1)^{k+1} & 2(-1)^k+1-(-1)^k\\ 0 & 1\end{pmatrix}=\begin{pmatrix}(-1)^{k+1} & 1-(-1)^{k+1}\\ 0 & 1\end{pmatrix}$

True for $n=1$; if true for $n=k$ then true for $n=k+1$; so true for all $n\in\mathbb{Z}^+$; $\mathbf{M}^{20}=\mathbf{I}$

D7 $n=1$: $1=1\times 1$. $k^2(2k^2-1)+(2k+1)^3=2k^4+8k^3+11k^2+6k+1=(k+1)^2\left(2(k+1)^2-1\right)$

True for $n=1$; if true for $n=k$ then true for $n=k+1$; so true for all $n\in\mathbb{Z}^+$; $n=10$: **19900**

D8 **(a)** $n=1$: $\mathbf{M}.\begin{pmatrix}1 & 1-2^k\\ 0 & 2^k\end{pmatrix}\mathbf{M}=\begin{pmatrix}1 & -1+2(1-2^k)\\ 0 & 2^{k+1}\end{pmatrix}=\begin{pmatrix}1 & 1-2^{k+1}\\ 0 & 2^{k+1}\end{pmatrix}$

True for $n=1$; if true for $n=k$ then true for $n=k+1$; so true for all $n\in\mathbb{Z}^+$

(b) $\det\mathbf{M}^n=2^n$; $3\times 2^n=192$: $n=6$

E Exam practice solutions

Mark-scheme notes are shown on the right of each line.

CORE *Exam questions for everyone*

E1 $n=1$: LHS $=3-3+1=1$, RHS $=1^3=1$ *B1*

Assume $\sum_{r=1}^{k}(3r^2-3r+1)=k^3$ *M1*

$\sum_{r=1}^{k+1}=k^3+3(k+1)^2-3(k+1)+1$ *M1*

$=k^3+3k^2+3k+1$ *A1*

$=(k+1)^3$ *A1*

True for $n=1$; if true for $n=k$ then true for $n=k+1$; so true for all $n\in\mathbb{Z}^+$ *A1*

E2 **(i)** $f(1)=81+8-9=80=16\times 5$ *B1*

$f(k+1)=9\times 3^{2k+2}+8k-1=9f(k)-64k+80$ *M1 A1*

$=9f(k)+16(5-4k)$; **True for $n=1$; if true for $n=k$ then true for $n=k+1$; so true for all $n\in\mathbb{Z}^+$** *A1*

(ii) $u_1=3-1=2$ *B1*

$u_{k+1}=3(3^k-1)+2=3^{k+1}-1$ *M1*

True for $n=1$; if true for $n=k$ then true for $n=k+1$; so true for all $n\in\mathbb{Z}^+$ *A1*

STRETCH

E3 **(a)** $n = 1$: $\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} = \mathbf{M}$ *B1*

Assume true for $n = k$: $\mathbf{M}^{k+1} = \begin{pmatrix} 2k+1 & -4k \\ k & 1-2k \end{pmatrix}\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$ *M1*

$= \begin{pmatrix} 6k+3-4k & -8k-4+4k \\ 3k+1-2k & -4k-1+2k \end{pmatrix}$ *A1*

$= \begin{pmatrix} 2(k+1)+1 & -4(k+1) \\ k+1 & 1-2(k+1) \end{pmatrix}$ *A1*

True for $n = 1$; if true for $n = k$ then true for $n = k+1$; so true for all $n \in \mathbb{Z}^+$ *A1*

(b) $\mathbf{M}^n \binom{1}{0} = \binom{2n+1}{n} = \binom{41}{20}$ *M1*

$n = 20$ *A1*

(c) $n = -1$: $\begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}$ *M1*

$\mathbf{M}\begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} -3+4 & 12-12 \\ -1+1 & 4-3 \end{pmatrix} = \mathbf{I}$, so it is $\mathbf{M}^{-1}$ *A1*