



eClassroom

A-LEVEL MATHEMATICS • PURE 2

Differentiation

Worked Solutions

Pearson Edexcel International A Level (WMA12) & A Level (9MA0) • Calculator allowed: show every step

Full worked solutions for the free Differentiation worksheet, A-Level and International A-Level Maths



Tap a section to jump straight to it

Full worked solutions for every question on the Pure 2 Differentiation worksheet, in the same order and with the same question numbers.

How to mark your work. Tick each step you got right, not just the final answer. At A-level most marks are for method, so a wrong final answer can still earn most of the marks. If you lost marks, find the *first* line where your working is different from ours. That is the thing to fix.

In Section E: **M** = method mark, **dM** = method mark that depends on the previous M mark, **A** = accuracy mark (needs the M mark before it), **B** = independent mark.

Question sheet

Exam questions by topic

Differentiation revision engine

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A**Your turn answers**

Compare each line of your working with the solution.

CORE

A1 $6x^2 + 6x - 36 = 6(x+3)(x-2) = 0$: $x = 2, y = -43$; $x = -3, y = 82$: $(2, -43), (-3, 82)$

A2 $1 - 9x^{-2} = 0, x = 3, y = 6$; $\frac{d^2y}{dx^2} = 18x^{-3} = \frac{2}{3} > 0$: $(3, 6)$, a minimum

A3 $f'(x) = 6x^2 - 18x = 6x(x-3) > 0$: outside the roots: $x < 0$ or $x > 3$

STRETCH

A4 $f'(x) = 3x^2 + 12x + 15 = 3(x+2)^2 + 3 \geq 3$, so $f'(x) > 0$ for all x

A5 $3x^{-\frac{1}{2}} - 1 = 0, x = 9, y = 18 - 9 = 9$; $\frac{d^2y}{dx^2} = -\frac{3}{2}x^{-\frac{3}{2}} < 0$: $(9, 9)$, a maximum

A6 Width x , length $\frac{64}{x}$: $P = 2x + \frac{128}{x}$; $2 - 128x^{-2} = 0, x = 8$; $\frac{d^2P}{dx^2} > 0$: 32 cm

A7 $3x^2 - 12 = 0$: stationary points $(-2, 17)$ and $(2, -15)$: $-15 < k < 17$



B**Fluency solutions**

One mark for each correct part. The notes in *italics* explain why each sequence was chosen.

CORE

B1 (a) $(3, -8)$ (b) $(2, 4)$ (c) $(3, -54)$, $(-3, 54)$ (d) $(0, 0)$, $(2, -16)$, $(-2, -16)$

Substitute each x into y , never into $\frac{dy}{dx}$

B2 (a) $2 > 0$: **minimum** (b) $-2 < 0$: **maximum** (c) $6x$: $(3, -54)$ **minimum**, $(-3, 54)$ **maximum**
(d) $12x^2 - 16$: $(0, 0)$ **maximum**, $(\pm 2, -16)$ **minima**

Positive second derivative: minimum; negative: maximum.

B3 (a) $x > 2$ (b) $x > 3$ (c) $3x^2 - 6x < 0$: $0 < x < 2$ (d) $3x^2 + 3 > 0$: **all x**

Solve the inequality for $f'(x)$, not for $f(x)$

B4 (a) $(2, 4)$ (b) $2x = 2x^{-2}$: $(1, 3)$ (c) $\frac{1}{2}x^{-\frac{1}{2}} = 1$: $\left(\frac{1}{4}, \frac{1}{4}\right)$ (d) $x^{-\frac{1}{2}} = 1$: $(1, -1)$

Write each term as a power of x before differentiating.

B5 (a) $y = \frac{20-x}{2}$, so $P = x \times \frac{20-x}{2} = 10x - \frac{1}{2}x^2$ **as required**

(b) $10 - x = 0$: $x = 10$ (c) $\frac{d^2P}{dx^2} = -1 < 0$: **greatest value 50**

B6 $f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$

(a) $f'(0) = 9$: **increasing** (b) $f'(2) = -3$: **decreasing** (c) $x = 1, x = 3$ (d) $1 < x < 3$

The sign of f' at one point shows the direction there; the roots split the regions.

STRETCH

B7 (a) $3x^2 + a = 0$ at $x = 2$: $a = -12$ (b) $2a + b = 0$ and $a + b = -3$: $a = 3, b = -6$

(c) $2x - kx^{-2} = 0$ at $x = 2$: $4 - \frac{k}{4} = 0$: $k = 16$ (d) $\frac{3}{2}kx^{\frac{1}{2}} - 6 = 0$ at $x = 4$: $3k = 6$: $k = 2$

A stationary point gives $\frac{dy}{dx} = 0$; a point on the curve gives a second equation.



B8 (a) $3(x+1)^2 + 3 > 0$ (b) $3(x-2)^2 + 3 > 0$ (c) $6\left(x + \frac{1}{2}\right)^2 + \frac{3}{2} > 0$ (d) $-3(x-1)^2 - 1 < 0$

Complete the square on $f'(x)$: the constant left over decides the sign.

B9 (a) $24x - 3x^2 = 0$, $x = 8$: **maximum 256** (b) $x = 10$: **maximum 200** (c) $3 = 75x^{-2}$, $x = 5$: **minimum 30** (d) $4x = 32x^{-2}$, $x = 2$: **minimum 24**

Each stationary value is found by putting x back into the quantity, then tested with the second derivative.

B10 (a) **3** (b) **2** (c) **1** (d) **1**

Three solutions between the stationary values, two at a stationary value, one outside.

C

Reasoning solutions

Your wording can be different. Check that your reason makes the same point.

CORE

C1 Sam has given the value of $\frac{dy}{dx}$, which is always 0 there. Put $x = 4$ into y : $16 - 32 + 3 = -13$: **(4, -13)**

C2 A nature must be justified by a test. $\frac{d^2y}{dx^2} = 6x = 12 > 0$ at $x = 2$: **a minimum**

C3 $\frac{dy}{dx} = \frac{15}{2}x^{\frac{1}{2}} - \frac{5}{2}x^{\frac{3}{2}} = \frac{5}{2}x^{\frac{1}{2}}(3 - x)$: dividing by $x^{\frac{1}{2}}$ loses $x = 0$
(0, 0) and **(3, $6\sqrt{3}$)**

C4 $f'(x) = 3(x-3)(x+1)$ is negative between its roots, not outside them: **$-1 < x < 3$**

STRETCH

C5 A second derivative of 0 does not decide the nature; the test fails. Look at the gradient $4x^3$ either side: negative for $x < 0$, positive for $x > 0$, so the curve goes down then up: **(0, 0) is a minimum**

C6 The greatest value can occur at a stationary point inside the interval. $f'(x) = 12 - 3x^2 = 0$, $x = 2$: $f(2) =$ **16**
 $16 > f(0) = 0$ and $16 > f(3) = 9$, so 16 is the greatest value.



C7 $f'(x) = 3x^2 - 3 = 3(x-1)(x+1) < 0$ for $-1 < x < 1$, so the graph slopes down there.

$y = f'(x)$ is a U-shaped parabola with vertex $(0, -3)$, crossing the x -axis at ± 1

C8 (a) $\frac{dy}{dx} = 2ax + b = 0$ at $x = -\frac{b}{2a}$; $\frac{d^2y}{dx^2} = 2a > 0$ when $a > 0$, so it is a minimum

as required

(b) $y = a \times \frac{b^2}{4a^2} - \frac{b^2}{2a} + c = c - \frac{b^2}{4a}$

as required

D

Problem solving solutions

There is often more than one correct method. Any complete, correct method scores full marks.

CORE

D1 $\frac{dy}{dx} = 3x^2 - 18x + 24 = 3(x-2)(x-4) = 0$

$x = 2$: $8 - 36 + 48 - 10 = 10$; $x = 4$: $64 - 144 + 96 - 10 = 6$: $A(2, 10)$, $B(4, 6)$

D2 (a) $\frac{dy}{dx} = 6x^{-\frac{1}{2}} - 2 = 0$, so $x^{\frac{1}{2}} = 3$: $x = 9$ only

(b) $\frac{d^2y}{dx^2} = -3x^{-\frac{3}{2}} = -\frac{1}{9} < 0$: a maximum

(c) $12\sqrt{x} = 2x$, so $\sqrt{x} = 6$: $Q(36, 0)$

D3 $f'(x) = 6x^2 - 6x - 36 = 6(x-3)(x+2) > 0$: outside the roots: $x < -2$ or $x > 3$

D4 (a) Fencing: $2x + L = 40$, so $L = 40 - 2x$ and $A = x(40 - 2x) = 40x - 2x^2$

as required

(b) $40 - 4x = 0$, $x = 10$, so the side parallel to the wall is 20 m; $\frac{d^2A}{dx^2} = -4 < 0$: 200 m^2 , a maximum

STRETCH

D5 (a) $x^2h = 500$, so $h = \frac{500}{x^2}$; $S = x^2 + 4xh = x^2 + \frac{2000}{x}$

as required

(b) $2x - 2000x^{-2} = 0$, $x^3 = 1000$, $x = 10$; $S = 100 + 200 = \text{300 cm}^2$, with height $h = 5 \text{ cm}$; $\frac{d^2S}{dx^2} = 2 + 4000x^{-3} = 6 > 0$, so a minimum



- D6** (a) Width $2x$, height $12 - x^2$: $A = 2x(12 - x^2) = 24x - 2x^3$ as required
- (b) $24 - 6x^2 = 0$, $x = 2$ ($x > 0$); $A = 48 - 16 = 32$, with $Q(2, 8)$; $\frac{d^2A}{dx^2} = -12x = -24 < 0$, so a maximum

- D7** (a) $\frac{dD}{dt} = \frac{3}{5}t^2 - 6t + 12 = 0$, so $t^2 - 10t + 20 = 0$, $t = 5 \pm \sqrt{5}$ as required
- (b) $\frac{d^2D}{dt^2} = \frac{6}{5}t - 6$; at $t = 5 - \sqrt{5}$ it is $-\frac{6\sqrt{5}}{5} < 0$, a maximum, with $D = 20 + 2\sqrt{5}$
- This exceeds the end values $D(0) = 10$ and $D(8) = 16.4$, so it is the greatest depth.

- D8** (a) $\frac{dy}{dx} = 4x^3 - 12x^2 = 4x^2(x - 3) = 0$: (0, 0), (3, -27)
- (b) $\frac{d^2y}{dx^2} = 12x^2 - 24x = 0$ at $x = 0$. $\frac{dy}{dx} < 0$ just either side of $x = 0$ ($4x^2 > 0$, $x - 3 < 0$), so the curve keeps going down: neither
- (c) $\frac{d^2y}{dx^2} = 108 - 72 = 36 > 0$ at $x = 3$: a minimum

E**Exam practice solutions**

Mark-scheme notes are shown on the right of each line.

CORE

Exam questions for everyone

- E1** (a) $\frac{dy}{dx} = 2x - 54x^{-2}$ B1
- (b) $2x = \frac{54}{x^2}$, so $x^3 = 27$, $x = 3$ M1
- $y = 9 + 18 = 27$: P(3, 27) A1
- (c) $\frac{d^2y}{dx^2} = 2 + 108x^{-3} = 6 > 0$, so P is a minimum B1
- (d) 0 < x < 3 B1



- E2** (a) $\pi r^2 h = 64\pi$, so $h = \frac{64}{r^2}$ B1
- $$S = \pi r^2 + 2\pi r h = \pi r^2 + 2\pi r \times \frac{64}{r^2}$$
- $$= \pi r^2 + \frac{128\pi}{r} \quad \text{as required} \quad \text{A1*}$$
- (b) $2\pi r - 128\pi r^{-2} = 0$, $r^3 = 64$: $r = 4$ B1
- (c) $S = 16\pi + 32\pi = 48\pi$ B1
- $$\frac{d^2 S}{dr^2} = 2\pi + 256\pi r^{-3} = 6\pi > 0$$
- , so a minimum
- B1

STRETCH

- E3** (a) $\frac{dy}{dx} = 4x^3 - 16x = 4x(x-2)(x+2) = 0$ M1
- $A(0, 3)$, $B(-2, -13)$, $D(2, -13)$ A1 A1
- (b) $\frac{d^2 y}{dx^2} = 12x^2 - 16$: $-16 < 0$ at A , a maximum; $32 > 0$ at B , a minimum B1 B1
- (c) A horizontal line between the minimum and the maximum values meets the curve four times:
 $-13 < k < 3$ M1 A1
- (d) $x < -2$ or $0 < x < 2$ B1

