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INTERNATIONAL A LEVEL MATHEMATICS • FURTHER PURE 3 (WFM03)

Further Pure 3 Practice Paper 1

Worked Solutions

AIMING FOR B

Written in the style of the Pearson Edexcel International Advanced Level paper (WFM03/01) • 75 marks

Every question solved in full on the paper itself, the way your teacher would write it out

How to read these solutions. Each page is the real question page, with a full answer written on the answer lines.

Red pen

M1 A1

boxed

the working a full-marks answer shows, with a short note on why each step is taken

the mark earned on that line: M = method, A = accuracy, B = independent mark

the final answer to each part

Sit the paper first. These solutions are most useful after you have tried every question yourself under timed conditions. Mark your paper with the mark scheme, then use this booklet to see exactly how each lost mark could have been earned.

Q	Topic	Marks	Page
1	A hyperbolic equation	6	2
2	A derivative and an exact integral	7	3
3	Integrals by completing the square	7	4
4	A 3×3 matrix with a parameter	8	5
5	The foci and directrices of an ellipse	8	6
6	A plane, a line and the angle between them	9	7
7	A reduction formula	9	8
8	Eigenvalues and diagonalisation	10	9
9	A tangent to a hyperbola and its asymptotes	11	10

Question paper & mark scheme | Exam questions by topic | More practice papers

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1. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

Solve the equation

$$2\cosh 2x - 9\cosh x + 7 = 0$$

giving your answers as exact logarithms where appropriate.

(6)

$\cosh 2x = 2\cosh^2 x - 1$: $2(2\cosh^2 x - 1) - 9\cosh x + 7 = 0$ M1

$4\cosh^2 x - 9\cosh x + 5 = 0$ A1

$(4\cosh x - 5)(\cosh x - 1) = 0 \Rightarrow \cosh x = \frac{5}{4}$ or $\cosh x = 1$ M1 A1

$\cosh x = 1 \Rightarrow x = 0$. $\cosh x = \frac{5}{4} \Rightarrow x = \pm \ln\left(\frac{5}{4} + \sqrt{\frac{25}{16} - 1}\right) = \pm \ln\left(\frac{5}{4} + \frac{3}{4}\right) = \pm \ln 2$ M1

$x = 0,\ x = \ln 2,\ x = -\ln 2$ (cosh is even, so both signs) A1

(Total for Question 1 is 6 marks)

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2.

$$y = x\,\text{arsinh}\,x - \sqrt{1+x^2}$$

(a) Show that $\dfrac{dy}{dx} = \text{arsinh}\,x$

(3)

(b) Hence find the exact value of

$$\int_0^{\frac{3}{4}} \text{arsinh}\,x\,dx$$

giving your answer in the form $a\ln 2 + b$, where a and b are rational numbers.

(4)

(a) $\frac{d}{dx}(x\,\text{arsinh}\,x) = \text{arsinh}\,x + x \times \frac{1}{\sqrt{1+x^2}}$ (product rule) M1

$\frac{d}{dx}\sqrt{1+x^2} = \frac{1}{2}(1+x^2)^{-1/2} \times 2x = \frac{x}{\sqrt{1+x^2}}$ M1

$\frac{dy}{dx} = \text{arsinh}\,x + \frac{x}{\sqrt{1+x^2}} - \frac{x}{\sqrt{1+x^2}} = \text{arsinh}\,x$ as required A1

(b) By (a), $\int \text{arsinh}\,x\,dx = x\,\text{arsinh}\,x - \sqrt{1+x^2}$: evaluate from 0 to $\frac{3}{4}$ M1

$\text{arsinh}\,\frac{3}{4} = \ln\left(\frac{3}{4} + \sqrt{\frac{9}{16}+1}\right) = \ln\left(\frac{3}{4} + \frac{5}{4}\right) = \ln 2$ M1

$= \left(\frac{3}{4}\ln 2 - \frac{5}{4}\right) - (0 - 1)$ A1

$(3/4)\ln 2 - 1/4$ A1

(Total for Question 2 is 7 marks)

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3. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

(a) Show that

$$\int_{-2}^{2} \frac{1}{\sqrt{x^2+4x+13}}\,\mathrm{d}x = \ln 3$$

(4)

(b) Find the exact value of

$$\int_{0}^{1} \frac{1}{4x^2+4x+5}\,\mathrm{d}x$$

(3)

(a) $x^2 + 4x + 13 = (x + 2)^2 + 9 = (x + 2)^2 + 3^2$ M1

$\int \frac{1}{\sqrt{(x + 2)^2 + 3^2}}\,dx = \text{arsinh}\left(\frac{x + 2}{3}\right)$ A1

$[\text{arsinh}\left(\frac{x + 2}{3}\right)]_{-2}^{2} = \text{arsinh}\ \frac{4}{3} - \text{arsinh}\ 0$ M1

$= \ln\left(\frac{4}{3} + \sqrt{\frac{16}{9} + 1}\right) - 0 = \ln\left(\frac{4}{3} + \frac{5}{3}\right) = \ln 3$ as required A1

(b) $4x^2 + 4x + 5 = (2x + 1)^2 + 4 = (2x + 1)^2 + 2^2$ M1

$\int \frac{1}{(2x + 1)^2 + 2^2}\,dx = \frac{1}{2} \times \frac{1}{2} \arctan\left(\frac{2x + 1}{2}\right) = \frac{1}{4} \arctan\left(\frac{2x + 1}{2}\right)$ A1

$(1/4)(\arctan(3/2) - \arctan(1/2))$ $\left(= \frac{1}{4} \arctan \frac{4}{7}\right)$ A1

(Total for Question 3 is 7 marks)

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4.

$$\mathbf{M} = \begin{pmatrix} 1 & 2 & k \\ 0 & k & 1 \\ 3 & 1 & 2 \end{pmatrix}$$

where k is a constant.

(a) Find $\det \mathbf{M}$ in terms of k

(2)

(b) Find the values of k for which $\mathbf{M}$ is singular.

(2)

Given that $k = 2$

(c) find $\mathbf{M}^{-1}$

(4)

(a) Expand along row 1: $1(2k - 1) - 2(0 - 3) + k(0 - 3k)$ M1

$= 2k - 1 + 6 - 3k^2 = \boxed{-3k^2 + 2k + 5}$ A1

(b) $-3k^2 + 2k + 5 = 0 \Rightarrow 3k^2 - 2k - 5 = 0 \Rightarrow (3k - 5)(k + 1) = 0$ M1

$\boxed{k = -1,\ k = 5/3}$ A1

(c) $k = 2$: $\det M = -3(4) + 2(2) + 5 = -3$ B1

Cofactors: $\begin{pmatrix} 3 & 3 & -6 \\ -2 & -4 & 5 \\ -2 & -1 & 2 \end{pmatrix}$ (signs + − + alternating) M1

Transpose, then divide by $\det M = -3$: M1

$M^{-1} = -\frac{1}{3}\begin{pmatrix} 3 & -2 & -2 \\ 3 & -4 & -1 \\ -6 & 5 & 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -3 & 2 & 2 \\ -3 & 4 & 1 \\ 6 & -5 & -2 \end{pmatrix}$ A1

Check: row 1 of M times column 1: $1(-1) + 2(-1) + 2(2) = 1$ ✓

(Total for Question 4 is 8 marks)

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5. The ellipse E has equation

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

(a) Find the eccentricity of E

(2)

(b) Find the coordinates of the foci of E and the equations of its directrices.

(3)

The point $P(4, p)$, where $p > 0$, lies on E. The foci of E are S and S'

(c) Find the lengths PS and PS', and verify that $PS + PS' = 10$

(3)

(a) $a = 5$, $b = 4$: $b^2 = a^2(1 - e^2) \Rightarrow 16 = 25(1 - e^2) \Rightarrow e^2 = \frac{9}{25}$ M1

$e = 3/5$ A1

(b) Foci $(\pm ae, 0) = (\pm 3, 0)$: $(3, 0)$ and $(-3, 0)$ B1

Directrices $x = \pm \frac{a}{e} = \pm 5 \div \frac{3}{5}$ M1

$x = 25/3$ and $x = -25/3$ A1

(c) $\frac{16}{25} + \frac{p^2}{16} = 1 \Rightarrow p^2 = \frac{144}{25} \Rightarrow p = \frac{12}{5}$ M1

$S(3, 0)$: $PS = \sqrt{1^2 + (\frac{12}{5})^2} = \sqrt{\frac{169}{25}} = \frac{13}{5}$; $S'(-3, 0)$: $PS' = \sqrt{7^2 + (\frac{12}{5})^2} = \frac{37}{5}$ M1

$PS + PS' = \frac{13}{5} + \frac{37}{5} = \frac{50}{5} = 10$ ✓ $(= 2a)$ A1

(Total for Question 5 is 8 marks)

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6. The plane Π contains the points $A(1, 0, 2)$, $B(3, 1, 1)$ and $C(2, -1, 4)$

(a) Find a Cartesian equation of Π

(4)

The line l has equation

$$\mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

(b) Find the coordinates of the point where l meets Π

(3)

(c) Find the acute angle between l and Π, giving your answer in degrees to one decimal place.

(2)

(a) $AB = (2, 1, -1)$, $AC = (1, -1, 2)$ M1

$AB \times AC = (1 \times 2 - (-1)(-1),\ (-1)(1) - 2 \times 2,\ 2(-1) - 1 \times 1)$ M1

$= (1, -5, -3)$ (check: $(1, -5, -3)\cdot(2, 1, -1) = 2 - 5 + 3 = 0$ ✓) A1

$x - 5y - 3z = d$, A: $d = 1 - 0 - 6 = -5$. $\boxed{x - 5y - 3z = -5}$ A1

(b) On l: $x = 2 + t$, $y = 2 - t$, $z = 1$. $(2 + t) - 5(2 - t) - 3(1) = -5$ M1

$6t - 11 = -5 \Rightarrow t = 1$ A1

$\boxed{(3, 1, 1)}$ (check: $3 - 5 - 3 = -5$ ✓) A1

(c) $\sin\theta = \dfrac{|(1, -5, -3)\cdot(1, -1, 0)|}{\sqrt{35}\sqrt{2}} = \dfrac{6}{\sqrt{70}}$ (sin, because n is perpendicular to the plane) M1

$\boxed{\theta = 45.8°}$ A1

(Total for Question 6 is 9 marks)

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7.

$$I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \, \mathrm{d}x, \qquad n \geqslant 0$$

(a) Show that, for $n \geqslant 2$,

$$I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}$$

(5)

(b) Hence find the exact value of I_4

(4)

(a) Parts: $u = x^n$, $dv/dx = \cos x$: $I_n = [x^n \sin x]_0^{\pi/2} - n\int_0^{\pi/2} x^{n-1} \sin x \, dx$ M1

$= (\frac{\pi}{2})^n - n\int_0^{\pi/2} x^{n-1} \sin x \, dx$ A1

Parts again: $\int x^{n-1} \sin x \, dx = [-x^{n-1} \cos x]_0^{\pi/2} + (n-1)\int x^{n-2} \cos x \, dx$ M1

At $\pi/2$, cos = 0; at 0, $x^{n-1} = 0$ ($n \geqslant 2$): boundary term = 0, leaving $(n-1)I_{n-2}$ A1

$I_n = (\frac{\pi}{2})^n - n(n-1)I_{n-2}$ as required A1

(b) $I_0 = \int_0^{\pi/2} \cos x \, dx = [\sin x]_0^{\pi/2} = 1$ B1

$I_2 = (\frac{\pi}{2})^2 - 2(1)I_0 = \frac{\pi^2}{4} - 2$ M1

$I_4 = (\frac{\pi}{2})^4 - 4(3)I_2 = \frac{\pi^4}{16} - 12(\frac{\pi^2}{4} - 2)$ M1

$\pi^4/16 - 3\pi^2 + 24$ A1

(Total for Question 7 is 9 marks)

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8.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & -2 \end{pmatrix}$$

(a) Show that 3 is an eigenvalue of $\mathbf{A}$, and find the other two eigenvalues.

(4)

(b) Find an eigenvector corresponding to each eigenvalue.

(3)

(c) Find an orthogonal matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{\mathrm{T}}\mathbf{AP} = \mathbf{D}$

(3)

(a) $\det(A - \lambda I)$: expand along row 2, which has only one non-zero entry:

$= (3 - \lambda)[(1 - \lambda)(-2 - \lambda) - 4] = (3 - \lambda)(\lambda^2 + \lambda - 6)$ M1

The factor $(3 - \lambda)$ shows $\lambda = 3$ is an eigenvalue A1

$\lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2) = 0$ M1

$\boxed{\lambda = 2, \ \lambda = -3}$ A1

(b) $\lambda = 3$: $-2x + 2z = 0$, $2x - 5z = 0 \Rightarrow x = z = 0$, y free: $(0, 1, 0)$ M1

$\lambda = 2$: $-x + 2z = 0$, $y = 0$: $(2, 0, 1)$ $\lambda = -3$: $4x + 2z = 0$, $y = 0$: $(1, 0, -2)$ A1

$\boxed{(0, 1, 0), \ (2, 0, 1), \ (1, 0, -2)}$ (check: $A(1, 0, -2) = (-3, 0, 6) = -3(1, 0, -2)$ ✓) A1

(c) Normalise: $|(2, 0, 1)| = |(1, 0, -2)| = \sqrt{5}$ M1

$P = \frac{1}{\sqrt{5}}\begin{pmatrix} 0 & 2 & 1 \\ \sqrt{5} & 0 & 0 \\ 0 & 1 & -2 \end{pmatrix}$ $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$ A1 A1

(The columns of P are in the same order as the eigenvalues in D.)

(Total for Question 8 is 10 marks)

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9.

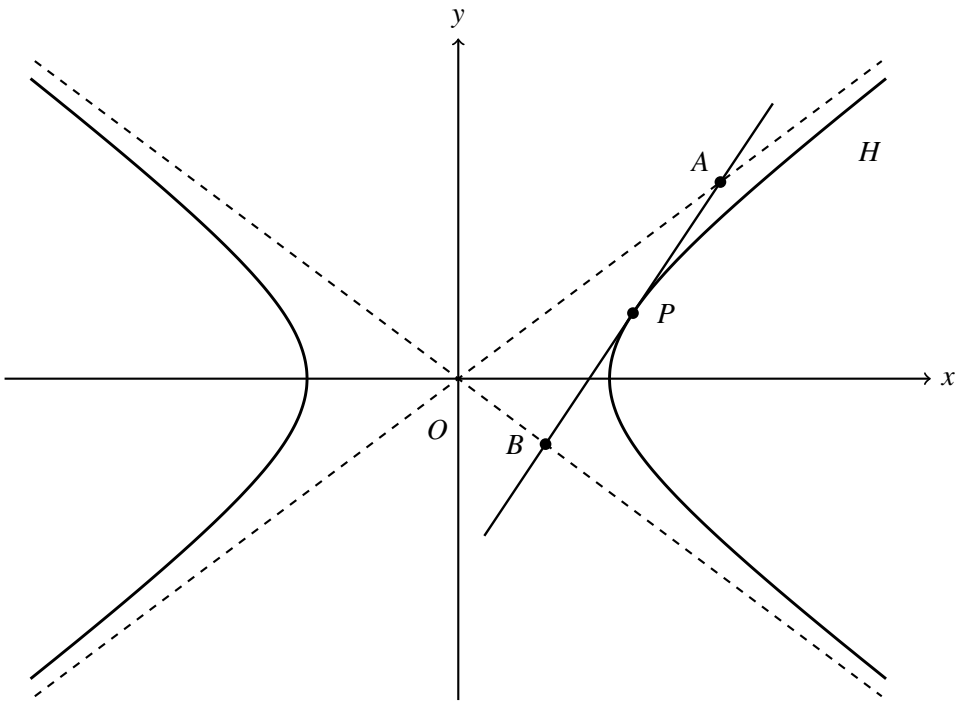

Figure 1

Figure 1 shows the hyperbola H with equation $\frac{x^2}{16} - \frac{y^2}{9} = 1$ and its asymptotes. The point $P(4\cosh t, 3\sinh t)$ lies on H. The tangent to H at P meets the asymptotes at A and B

(a) Show that an equation of the tangent to H at P is

$$3x\cosh t - 4y\sinh t = 12$$

(4)

(b) Find, in terms of t, the coordinates of A and B, giving your answers in terms of e^t

(4)

(c) Show that the area of the triangle OAB, where O is the origin, does not depend on t

(3)

(a) $\frac{dx}{dt} = 4\sinh t$, $\frac{dy}{dt} = 3\cosh t$, so $\frac{dy}{dx} = \frac{3\cosh t}{4\sinh t}$ M1 A1

$y - 3\sinh t = \frac{3\cosh t}{4\sinh t}(x - 4\cosh t)$ M1

$\times 4\sinh t$: $4y\sinh t - 12\sinh^2 t = 3x\cosh t - 12\cosh^2 t$

$3x\cosh t - 4y\sinh t = 12(\cosh^2 t - \sinh^2 t) = 12$ as required A1

(b) Asymptotes $y = \pm\frac{3}{4}x$. $y = \frac{3}{4}x$: $3x\cosh t - 3x\sinh t = 12 \Rightarrow x(\cosh t - \sinh t) = 4$ M1

$\cosh t - \sinh t = e^{-t}$, so $x = 4e^t$, $y = 3e^t$; similarly $\cosh t + \sinh t = e^t$ M1

$A(4e^t, 3e^t)$ A1

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Question 9 continued

$B(4e^{-t}, -3e^{-t})$ A1

(c) Area of $OAB = \frac{1}{2}|x_A y_B - x_B y_A|$ M1

$= \frac{1}{2}|4e^{t}(-3e^{-t}) - 4e^{-t}(3e^{t})|$ M1

$= \frac{1}{2}|-12 - 12| = 12$, which does not depend on t A1

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(Total for Question 9 is 11 marks)

TOTAL FOR PAPER IS 75 MARKS

END

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