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INTERNATIONAL A LEVEL MATHEMATICS • FURTHER PURE MATHEMATICS 1

Further Pure 1

Chapter 8: Proof

Exam Questions by Topic

Pearson Edexcel International Advanced Level (WFM01/01) • Calculator allowed

Multi-step exam questions at the difficulty of the real WFM01 paper, with a full mark scheme

Chapters (tap a chapter to jump to it)		Qs	Marks	Page	My score
1	**Complex Numbers** Arithmetic, conjugates, Argand diagrams, modulus and argument, roots of polynomial equations	10	80	separate file	
2	**Roots of Quadratic Equations** Sums and products of roots, symmetric functions, equations with transformed roots	10	80	separate file	
3	**Numerical Solutions of Equations** Interval bisection, linear interpolation, Newton–Raphson	10	80	separate file	
4	**Coordinate Systems** Parametric equations, parabolas, rectangular hyperbolas, tangents and normals	10	80	separate file	
5	**Matrices** Matrix arithmetic, determinants, inverses of 2×2 matrices	10	80	separate file	
6	**Transformations Using Matrices** Linear transformations, combined transformations, area scale factor, inverses	10	80	separate file	
7	**Series** Standard sums of r, r^2 and r^3, sigma notation	10	80	separate file	
8	**Proof** Mathematical induction for sums, divisibility, matrices and sequences	10	80	2	

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CHAPTER 8 • KEY SKILLS

8 Proof

Specification FP1: Proof by induction • 10 questions • 80 marks • about 96 minutes

1 Four parts

Basis ($n = 1$), assumption ($n = k$), inductive step ($n = k+1$), conclusion. Every part earns marks.

2 The basis case

Show both sides for $n = 1$ with working, e.g. "LHS = 1, RHS = $\frac{1}{3}(1)(1)(3) = 1$".

3 Sums

Add the $(k+1)$th term to the assumed sum, then factorise towards the formula with $k+1$ in place of n.

4 Divisibility

Write f$(k+1)$ in terms of f(k), e.g. f$(k+1)$ = f$(k)+5\times 6^k$, so it is a sum of multiples.

5 Matrix powers

$\mathbf{M}^{k+1} = \mathbf{M}^k\mathbf{M}$ (or $\mathbf{M}\mathbf{M}^k$). Multiply out and simplify each entry to the $n = k+1$ form.

6 Recurrence relations

Substitute the assumed u_k into $u_{k+1} = \ldots$ and simplify to the formula with $n = k+1$.

7 Index laws

$6^{k+1} = 6\times 6^k$, $5^{2(k+1)} = 25\times 5^{2k}$, $2^{k+3} = 2\times 2^{k+2}$

8 The conclusion

"If true for $n = k$ then true for $n = k+1$; true for $n = 1$; so true for all positive integers n."

9 Using the result

After the proof, substitute into the formula: do not multiply matrices out again.

Skills map every skill above is tested at least once; interleaved skills come from GCSE or other chapters

Skill	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10
Marks	6	6	6	6	6	10	10	10	10	10
1 Four parts	●	●	●	●	●	●	●	●	●	●
2 The basis case	●	●	●	●	●	●	●	●	●	●
3 Sums	●				●			●		
4 Divisibility		●					●			
5 Matrix powers			●			●				●
6 Recurrence relations				●					●	
7 Index laws		●	●	●		●	●	●	●	
8 The conclusion	●	●	●	●	●	●	●	●	●	●
9 Using the result						●	●	●	●	●
Interleaved: series (Chapter 7)	●				●			●		
Interleaved: matrices and transformations (Chapters 5, 6)			●			●				●
Interleaved: indices, logarithms and sequences (P1, P2)		●		●			●	●	●	

Exam difficulty. Questions 1 to 5 check each skill on its own. Questions 6 to 10 are multi-step questions built like the last questions on a real WFM01 paper, where one part leads into the next and skills from different topics are combined.

Must learn. Show the $n = 1$ case on both sides, state the assumption for $n = k$, show the algebra to the $n = k+1$ form, and finish with the full conclusion every time.

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1. Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\sum_{r=1}^{n}(2r-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$$

(6)

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Question 1 continued

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(Total for Question 1 is 6 marks)

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2. Use mathematical induction to show that $6^n + 4$ is a multiple of 5 for every positive integer n.

(6)

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Question 2 continued

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(Total for Question 2 is 6 marks)

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3.

$$\mathbf{M} = \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}$$

Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\mathbf{M}^n = \begin{pmatrix} 3^n & 0 \\ 3^n - 1 & 1 \end{pmatrix}$$

(6)

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Question 3 continued

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4. A sequence of numbers is defined by

$$u_1 = 2, \qquad u_{n+1} = 3u_n - 2, \quad n \geqslant 1$$

Prove by induction that, for $n \in \mathbb{Z}^+$, $u_n = 3^{n-1} + 1$

(6)

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Question 4 continued

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5. A student claims that, for all positive integers n,

$$1+2+3+\cdots+n=\frac{1}{2}\left(n^2+n+2\right)$$

(a) Show that, if the claim is true for $n=k$, then it is true for $n=k+1$

(3)

(b) Show that the claim is false.

(2)

(c) Explain why part (a) does not prove the claim.

(1)

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Question 5 continued

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(Total for Question 5 is 6 marks)

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6.

$$\mathbf{M} = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$$

(a) Prove by induction that, for $n \in \mathbb{Z}^+$, $\mathbf{M}^n = \begin{pmatrix} 2^n & 1-2^n \\ 0 & 1 \end{pmatrix}$

(5)

(b) Hence write down $\mathbf{N} = \mathbf{M}^5$

(1)

The matrix $\mathbf{B} = \mathbf{NT}$, where $\mathbf{T} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$. A region R is transformed by $\mathbf{B}$ onto a region of area 480

(c) Find the area of R.

(3)

(d) Show that the formula in part (a) also gives the correct matrix when $n = 0$

(1)

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Question 6 continued

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Question 6 continued

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(Total for Question 6 is 10 marks)

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7.

$$f(n) = 5^{2n} - 1$$

(a) Show that $f(k+1) = 25f(k) + 24$, and hence prove by induction that $f(n)$ is divisible by 24 for all $n \in \mathbb{Z}^+$

(6)

(b) Hence find the remainder when 5^{2n+1} is divided by 24

(2)

(c) Determine whether $5^n - 1$ is divisible by 24 for all $n \in \mathbb{Z}^+$

(2)

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Question 7 continued

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Question 7 continued

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(Total for Question 7 is 10 marks)

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8.

(a) Prove by induction that, for $n \in \mathbb{Z}^+$, $\sum_{r=1}^{n} r \times 2^r = 2 + (n-1)2^{n+1}$

(6)

(b) Hence find the value of $\sum_{r=1}^{10} r \times 2^r$

(2)

(c) Hence find the value of $\sum_{r=11}^{20} r \times 2^r$

(2)

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Question 8 continued

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(Total for Question 8 is 10 marks)

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9. A sequence is defined by

$$u_1 = 5, \qquad u_{n+1} = 2u_n + 3, \quad n \geqslant 1$$

(a) Find u_2 and u_3.

(2)

(b) Prove by induction that, for $n \in \mathbb{Z}^+$, $u_n = 2^{n+2} - 3$

(5)

(c) Find the smallest value of n for which $u_n > 1000$

(3)

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Question 9 continued

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Question 9 continued

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10.

$$\mathbf{M} = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$$

(a) Prove by induction that, for $n \in \mathbb{Z}^+$, $\mathbf{M}^n = \begin{pmatrix} 2n+1 & -2n \\ 2n & 1-2n \end{pmatrix}$

(5)

(b) Show that $\det(\mathbf{M}^n) = 1$ for all n, and state what this means for the areas of shapes transformed by $\mathbf{M}^n$

(2)

(c) Show that the point (1, 1) is mapped to itself by $\mathbf{M}^n$ for every positive integer n, and find the image of the point (1, 0) under $\mathbf{M}^{10}$

(3)

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TOTAL FOR CHAPTER 8 IS 80 MARKS

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