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INTERNATIONAL A LEVEL MATHEMATICS • PURE 3 (WMA13)

Pure 3 Practice Paper 2

Worked Solutions

AIMING FOR A

Written in the style of the Pearson Edexcel International Advanced Level paper (WMA13/01) • 75 marks

Every question solved in full on the paper itself, the way your teacher would write it out

How to read these solutions. Each page is the real question page, with a full answer written on the answer lines.

Red pen

M1 A1

boxed

the working a full-marks answer shows, with a short note on why each step is taken

the mark earned on that line: M = method, A = accuracy, B = independent mark

the final answer to each part

Sit the paper first. These solutions are most useful after you have tried every question yourself under timed conditions. Mark your paper with the mark scheme, then use this booklet to see exactly how each lost mark could have been earned.

Q	Topic	Marks	Page
1	Inverse function, sketch and a root	10	2
2	Double angle identity and equation	8	4
3	Logarithmic graph of a model	8	5
4	Differentiating arctan using $\frac{dx}{dy}$	8	7
5	Area using a trigonometric identity	8	9
6	Exponential model, maximum and rate	8	11
7	Modulus of an exponential function	9	13
8	Differentiating and integrating $\sec x$	8	14
9	Rational function and its inverse	8	15

Question paper & mark scheme **Exam questions by topic** **More practice papers**

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1. The function f is defined by

$$f(x) = e^{2x} - 4 \qquad x \in \mathbb{R}$$

(a) Find $f^{-1}(x)$, stating its domain.

(3)

(b) On the same axes, sketch the curves with equations $y = f(x)$ and $y = f^{-1}(x)$

On your sketch, show the coordinates of each point where each curve meets the coordinate axes, and the equation of each asymptote.

(3)

The curves $y = f(x)$ and $y = f^{-1}(x)$ meet at a point with positive x coordinate α.

(c) Explain why α is a root of the equation

$$e^{2x} - x - 4 = 0$$

(2)

(d) Show that $\alpha = 0.7825$ to 4 decimal places.

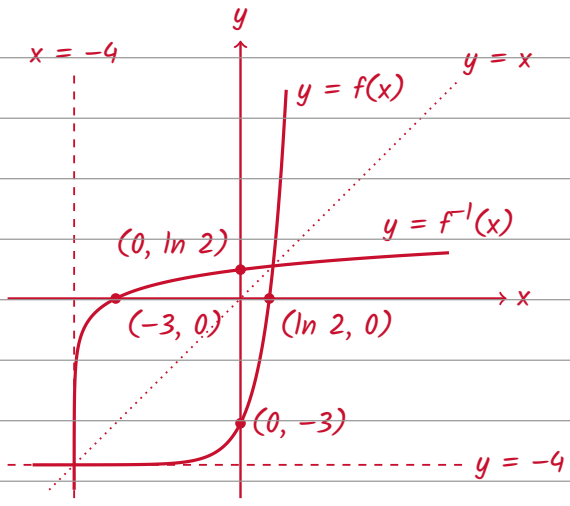
(2)

(a) $y = e^{2x} - 4 \Rightarrow e^{2x} = y + 4 \Rightarrow 2x = \ln(y + 4) \Rightarrow x = \frac{1}{2}\ln(y + 4)$ M1

$f^{-1}(x) = \frac{1}{2}\ln(x + 4)$ A1

Domain of f^{-1} = range of f: $e^{2x} > 0$, so $f(x) > -4$ $x > -4$ B1

(b) f: (0, −3), $e^{2x} = 4$ at $x = \frac{1}{2}\ln 4 = \ln 2$, asymptote $y = -4$. f^{-1}: reflect in $y = x$. B1 B1 B1



(c) $y = f^{-1}(x)$ is the reflection of $y = f(x)$ in the line $y = x$, and f is increasing, so the curves meet on the line $y = x$: $f(\alpha) = \alpha$ M1

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Question 1 continued

$e^{2x} - 4 = x \Rightarrow e^{2x} - x - 4 = 0$ ✓ A1

(d) $g(x) = e^{2x} - x - 4$. Test the bounds for 4 d.p.: 0.78245 and 0.78255

$g(0.78245) = -0.00025... < 0$ $g(0.78255) = 0.00060... > 0$ M1

Sign change and g is continuous, so $0.78245 < \alpha < 0.78255$ $\alpha = 0.7825$ (4 d.p.) A1

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(Total for Question 1 is 10 marks)

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2. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

(a) Express $\cot\theta - \tan\theta$ in the form $k\cot 2\theta$, where k is a constant to be found.

(3)

(b) Hence solve, for $0 < \theta < \pi$, the equation

$$\cot\theta - \tan\theta = 4$$

giving your answers to 3 decimal places.

(3)

(c) Use the result of part (a) to find the exact value of $\tan\left(\frac{\pi}{12}\right)$, giving your answer in the form $a + b\sqrt{3}$, where a and b are integers.

(2)

(a) $\cot\theta - \tan\theta = \frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta} = \frac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta}$ M1

$= \cos 2\theta \div (\frac{1}{2}\sin 2\theta) = \frac{2\cos 2\theta}{\sin 2\theta}$ (double angle formulae) M1

$\cot\theta - \tan\theta = 2\cot 2\theta$, so $k = 2$ A1

(b) $2\cot 2\theta = 4 \Rightarrow \cot 2\theta = 2 \Rightarrow \tan 2\theta = \frac{1}{2}$ M1

$0 < \theta < \pi$, so $0 < 2\theta < 2\pi$: $2\theta = 0.46364\ldots$ or $\pi + 0.46364\ldots = 3.60524\ldots$

$\theta = 0.232$ $\theta = 1.803$ A1 A1

(c) $\theta = \frac{\pi}{12}$: $\cot\frac{\pi}{12} - \tan\frac{\pi}{12} = 2\cot\frac{\pi}{6} = 2\sqrt{3}$. Let $t = \tan\frac{\pi}{12}$.

$\frac{1}{t} - t = 2\sqrt{3} \Rightarrow 1 - t^2 = 2\sqrt{3}t \Rightarrow t^2 + 2\sqrt{3}t - 1 = 0$ M1

$t = -\sqrt{3} \pm \sqrt{3+1} = -\sqrt{3} \pm 2$, and $\tan\frac{\pi}{12} > 0$: $\tan(\pi/12) = 2 - \sqrt{3}$ A1

(Total for Question 2 is 8 marks)

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3. The resting energy use, R kJ per hour, of a mammal of mass m kg is modelled by the equation

$$R = a m^n$$

where a and n are constants.

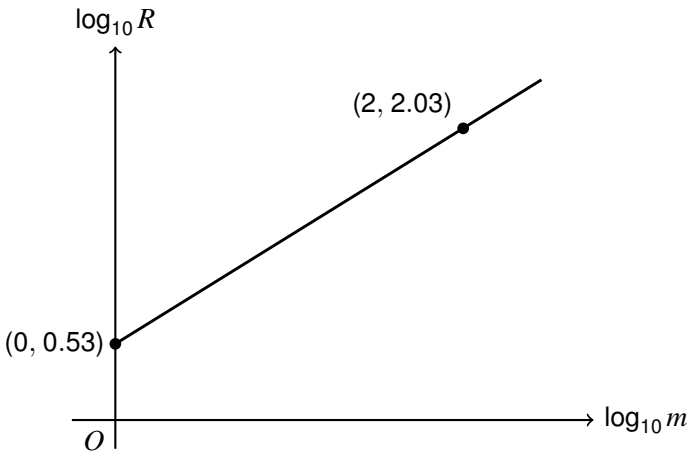

Figure 1

Figure 1 shows the linear relationship between $\log_{10} R$ and $\log_{10} m$

The line passes through the points (0, 0.53) and (2, 2.03)

(a) Find, according to the model, the value of n and the value of a, giving a to 3 significant figures.

(4)

(b) Use the model to estimate the resting energy use of a mammal of mass 70 kg.

(1)

(c) Find, according to the model, the mass of a mammal for which $\frac{dR}{dm} = 1$

(3)

(a) Take logs: $\log R = \log a + n \log m$, a straight line with gradient n and intercept $\log a$ M1

$n = \frac{2.03 - 0.53}{2 - 0} = \frac{1.5}{2}$ $\boxed{n = 0.75}$ B1

$\log a = 0.53 \Rightarrow a = 10^{0.53} = 3.3884...$ M1

$\boxed{a = 3.39}$ A1

(b) $R = 3.3884... \times 70^{0.75} = 82.00...$ $\boxed{82.0 \text{ kJ per hour}}$ B1

(c) $\frac{dR}{dm} = 0.75 \times 3.3884... \times m^{-0.25} = 2.5413... \times m^{-0.25}$ M1

$2.5413... \times m^{-0.25} = 1 \Rightarrow m^{0.25} = 2.5413... \Rightarrow m = (2.5413...)^4$ M1

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Question 3 continued

$m = 41.7$ kg A1

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(Total for Question 3 is 8 marks)

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4.

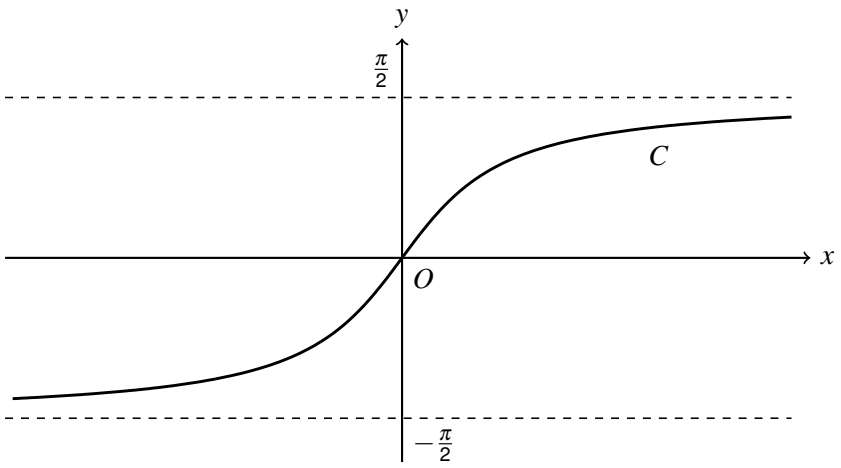

Figure 2

Figure 2 shows a sketch of the curve C with equation

$$y = \arctan\left(\frac{x}{2}\right) \qquad x \in \mathbb{R}$$

(a) Use $x = 2\tan y$ to find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ as a function of x, giving your answer in its simplest form.

(3)

The point A on C has x coordinate 2

The normal to C at A meets the x-axis at the point B.

(b) Find the exact x coordinate of B.

(3)

(c) Find the exact coordinates of each point on C at which the gradient of C is $\dfrac{1}{5}$

(2)

(a) $x = 2\tan y \Rightarrow \frac{dx}{dy} = 2\sec^2 y = 2(1 + \tan^2 y)$ M1

$\tan y = \frac{x}{2}$: $\frac{dx}{dy} = 2\left(1 + \frac{x^2}{4}\right) = \frac{4 + x^2}{2}$ M1

$dy/dx = 2/(4 + x^2)$ A1

(b) $x = 2$: $y = \arctan 1 = \frac{\pi}{4}$, gradient $= \frac{2}{8} = \frac{1}{4}$ M1

Normal gradient $= -4$: $y - \frac{\pi}{4} = -4(x - 2)$ M1

At B, $y = 0$: $\frac{\pi}{4} = 4(x - 2) \Rightarrow x - 2 = \frac{\pi}{16}$ $x = 2 + \pi/16$ A1

(c) $\frac{2}{4 + x^2} = \frac{1}{5} \Rightarrow 4 + x^2 = 10 \Rightarrow x^2 = 6 \Rightarrow x = \pm\sqrt{6}$ (two points) M1

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Question 4 continued

$(\sqrt{6}, \arctan(\sqrt{6}/2))$ and $(-\sqrt{6}, -\arctan(\sqrt{6}/2))$ A1

(Total for Question 4 is 8 marks)

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5. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

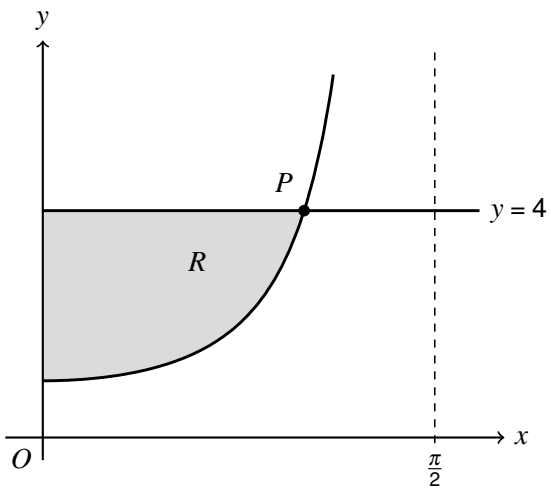

Figure 3

Figure 3 shows part of the curve C with equation

$$y = \frac{2}{1+\cos 2x} \qquad 0 \leqslant x < \frac{\pi}{2}$$

and the line with equation $y = 4$

The line meets C at the point P.

The region R, shown shaded in Figure 3, is bounded by C, the line $y = 4$ and the y-axis.

(a) Find the exact x coordinate of P.

(2)

(b) Find the exact area of R, giving your answer in the form $p\pi + q\sqrt{3}$, where p and q are rational constants.

(6)

(a) $\frac{2}{1+\cos 2x} = 4 \Rightarrow 1 + \cos 2x = \frac{1}{2} \Rightarrow \cos 2x = -\frac{1}{2}$ M1

$0 \leqslant 2x < \pi$: $2x = \frac{2\pi}{3}$ $\boxed{x = \pi/3}$ A1

(b) $\cos 2x = 2\cos^2 x - 1$, so $1 + \cos 2x = 2\cos^2 x$ M1

$y = \frac{2}{2\cos^2 x} = \sec^2 x$ (so C is $y = \sec^2 x$, which we can integrate) A1

R = (area under $y = 4$) − (area under C), from $x = 0$ to $x = \frac{\pi}{3}$: $\int (4 - \sec^2 x)\, dx$ M1

$\int (4 - \sec^2 x)\, dx = 4x - \tan x$ M1

$[4x - \tan x]$ from 0 to $\frac{\pi}{3}$ $= (\frac{4\pi}{3} - \sqrt{3}) - (0 - 0)$ M1

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Question 5 continued

Area of $R = 4\pi/3 - \sqrt{3}$ $(p = \frac{4}{3},\ q = -1)$ A1

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(Total for Question 5 is 8 marks)

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6. A drug is given to a patient. The concentration of the drug in the patient's blood, C mg per litre, t hours after it is given, is modelled by the equation

$$C = 12\left(e^{-0.2t} - e^{-0.8t}\right) \qquad t \geqslant 0$$

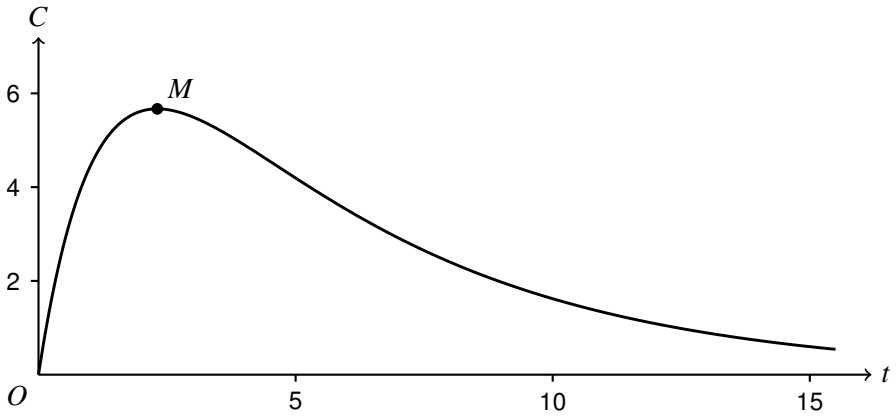

Figure 4

Figure 4 shows the graph of C against t. The concentration is greatest at the point M.

(a) Find the exact value of t at M, giving your answer in the form $p \ln q$, where p and q are rational.

(3)

(b) Find the greatest concentration of the drug, according to the model.

(2)

The drug is effective only while $C \geqslant 2$

(c) Show that, after the greatest concentration is reached, the drug stops being effective between $t = 8$ and $t = 9$

(2)

(d) Find the rate of change of the concentration when $t = 10$, stating the units of your answer.

(1)

(a) $\frac{dC}{dt} = 12(-0.2e^{-0.2t} + 0.8e^{-0.8t})$ M1

At M, $\frac{dC}{dt} = 0$: $0.8e^{-0.8t} = 0.2e^{-0.2t} \Rightarrow 4 = e^{0.6t} \Rightarrow 0.6t = \ln 4$ M1

$t = (5/3)\ln 4$ $(= 2.31\ldots)$ A1

(b) $e^{-0.2t} = e^{-(1/3)\ln 4} = 4^{-1/3}$, $e^{-0.8t} = 4^{-4/3} = \frac{1}{4} \times 4^{-1/3}$ M1

$C = 12 \times 4^{-1/3} \times (1 - \frac{1}{4}) = 9 \times 4^{-1/3} = 5.669\ldots$ 5.67 mg per litre A1

(c) $C(8) = 12(e^{-1.6} - e^{-6.4}) = 2.40\ldots > 2$ $C(9) = 12(e^{-1.8} - e^{-7.2}) = 1.97\ldots < 2$ M1

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Question 6 continued

C is continuous and falling after M, so it drops below 2 between $t = 8$ and $t = 9$ ✓ A1

(d) $\frac{dC}{dt} = 12(-0.2e^{-2} + 0.8e^{-8}) = -0.3215...$ -0.322 mg per litre per hour B1

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(Total for Question 6 is 8 marks)

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7. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

(a) Sketch the graph with equation

$$y = |e^x - 4|$$

showing the coordinates of each point where the graph meets the coordinate axes, and the equation of the asymptote.

(3)

(b) Solve the equation $|e^x - 4| = 3$

(2)

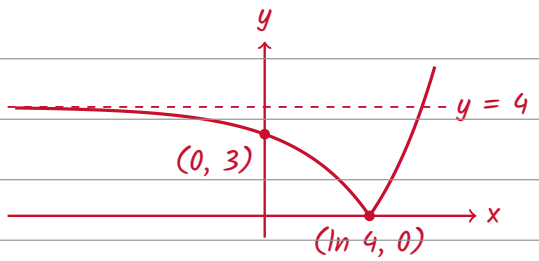
(c) Find the exact solutions of the equation

$$|e^x - 4| = 2e^{-x}$$

(4)

(a) $y = e^x - 4$ meets the axes at $(0, -3)$ and $(\ln 4, 0)$, with asymptote $y = -4$.

Reflect the part below the x-axis: $(0, -3)$ becomes $(0, 3)$ and $y = -4$ becomes $y = 4$. B1 B1 B1



(b) $e^x - 4 = 3$ or $e^x - 4 = -3 \Rightarrow e^x = 7$ or $e^x = 1$ M1

$x = \ln 7$ or $x = 0$ A1

(c) Case $e^x \geqslant 4$: $e^x - 4 = 2e^{-x}$. Multiply by e^x: $e^{2x} - 4e^x - 2 = 0$

$e^x = 2 \pm \sqrt{6}$: $2 - \sqrt{6} < 0$ (reject), $2 + \sqrt{6} = 4.45\ldots \geqslant 4$ ✓ M1

$x = \ln(2 + \sqrt{6})$ A1

Case $e^x < 4$: $4 - e^x = 2e^{-x} \Rightarrow e^{2x} - 4e^x + 2 = 0 \Rightarrow e^x = 2 \pm \sqrt{2}$ M1

Both positive and below 4 ✓ $x = \ln(2 + \sqrt{2})$ or $x = \ln(2 - \sqrt{2})$ A1

(Total for Question 7 is 9 marks)

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8. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

(a) Given that

$$y = \ln(\sec x + \tan x) \qquad 0 \leqslant x < \frac{\pi}{2}$$

find $\frac{dy}{dx}$, giving your answer in its simplest form.

(3)

(b) Hence find the exact value of

$$\int_0^{\frac{\pi}{3}} \sec x \, dx$$

(2)

(c) Find the exact value of

$$\int_0^{\frac{\pi}{3}} (\sec x + \tan x)^2 \, dx$$

(3)

(a) $y = \ln u$ with $u = \sec x + \tan x$: $\frac{du}{dx} = \sec x \tan x + \sec^2 x$ M1

$\frac{dy}{dx} = \frac{1}{u} \times \frac{du}{dx} = (\sec x \tan x + \sec^2 x) \div (\sec x + \tan x)$ A1

$= \sec x(\tan x + \sec x) \div (\sec x + \tan x)$ $\boxed{dy/dx = \sec x}$ A1

(b) So $\int \sec x \, dx = \ln(\sec x + \tan x)$. At $\frac{\pi}{3}$: sec = 2, tan = $\sqrt{3}$. At 0: sec = 1, tan = 0.

$[\ln(\sec x + \tan x)]$ from 0 to $\frac{\pi}{3}$ $= \ln(2 + \sqrt{3}) - \ln 1$ M1

$\boxed{\ln(2 + \sqrt{3})}$ A1

(c) $(\sec x + \tan x)^2 = \sec^2 x + 2\sec x \tan x + \tan^2 x = 2\sec^2 x + 2\sec x \tan x - 1$ M1

$\int (2\sec^2 x + 2\sec x \tan x - 1) \, dx = 2\tan x + 2\sec x - x$ M1

$(2\sqrt{3} + 4 - \frac{\pi}{3}) - (0 + 2 - 0)$ $\boxed{2\sqrt{3} + 2 - \pi/3}$ A1

(Total for Question 8 is 8 marks)

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9. The function f is defined by

$$f(x) = \frac{2x^2 + 3x - 2}{x^2 - 4} \qquad x \in \mathbb{R},\ x \neq -2,\ x \neq 2$$

(a) Write f(x) as a fraction in its simplest form.

(2)

(b) Find $f^{-1}(x)$, stating its domain.

(3)

(c) Find the coordinates of each point on the curve $y = f(x)$ at which the gradient is -3

(3)

(a) $2x^2 + 3x - 2 = (2x - 1)(x + 2)$ and $x^2 - 4 = (x - 2)(x + 2)$ M1

Cancel the common factor $(x + 2)$: $\boxed{f(x) = (2x - 1)/(x - 2)}$ A1

(b) $y = \frac{2x - 1}{x - 2} \Rightarrow xy - 2y = 2x - 1 \Rightarrow xy - 2x = 2y - 1 \Rightarrow x(y - 2) = 2y - 1$ M1

$\boxed{f^{-1}(x) = (2x - 1)/(x - 2)}$ (the same rule as f) A1

Domain of f^{-1} = range of f. $f(x) \neq 2$, and $x \neq -2$ also removes $\frac{2(-2) - 1}{-2 - 2} = \frac{5}{4}$

$\boxed{x \neq 2,\ x \neq 5/4}$ B1

(c) Quotient rule: $f'(x) = \frac{2(x - 2) - (2x - 1)}{(x - 2)^2} = \frac{-3}{(x - 2)^2}$ M1

$\frac{-3}{(x - 2)^2} = -3 \Rightarrow (x - 2)^2 = 1 \Rightarrow x - 2 = \pm 1 \Rightarrow x = 1$ or $x = 3$ M1

$f(1) = \frac{1}{-1} = -1$, $f(3) = \frac{5}{1} = 5$ $\boxed{(1, -1) \text{ and } (3, 5)}$ A1

(Total for Question 9 is 8 marks)

TOTAL FOR PAPER IS 75 MARKS

END

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