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A-LEVEL MATHEMATICS • FURTHER PURE 3

Differentiation

Worked Solutions

Pearson Edexcel International A Level (WFM03) • Calculator allowed: show every step

Full worked solutions for the free Differentiation worksheet, A-Level and International A-Level Maths

A **Examples** & Your turn page 2

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Tap a section to jump straight to it

Full worked solutions for every question on the Further Pure 3 Differentiation worksheet, in the same order and with the same question numbers.

How to mark your work. Tick each step you got right, not just the final answer. At A-level most marks are for method, so a wrong final answer can still earn most of the marks. If you lost marks, find the *first* line where your working is different from ours. That is the thing to fix.

In Section E: **M** = method mark, **dM** = method mark that depends on the previous M mark, **A** = accuracy mark (needs the M mark before it), **B** = independent mark.

Question sheet | Exam questions by topic | Further Pure 3 exam questions | More worksheets

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A Your turn answers

Compare each line of your working with the solution.

CORE

A1 $\operatorname{sech}^2(x^2) \times 2x$: $\boxed{2x\operatorname{sech}^2(x^2)}$

A2 $\boxed{e^{2x}(2\cosh x + \sinh x)}$

A3 $\dfrac{1}{\sqrt{1-\frac{x^2}{9}}} \times \dfrac{1}{3}$: $\boxed{\dfrac{1}{\sqrt{9-x^2}}}$

STRETCH

A4 $\dfrac{1}{1-\frac{x^2}{4}} \times \dfrac{1}{2}$: $\boxed{\dfrac{2}{4-x^2}}$

A5 $x = \cos y$, $\dfrac{dx}{dy} = -\sin y = -\sqrt{1-x^2}$ (for $0 < y < \pi$), so $\dfrac{dy}{dx} = -\dfrac{1}{\sqrt{1-x^2}}$, as required

A6 $4\sinh x - 2\cosh x = 0$, $\tanh x = \dfrac{1}{2}$, $x = \dfrac{1}{2}\ln 3$: $\boxed{\left(\dfrac{1}{2}\ln 3,\ 2\sqrt{3}\right)}$

A7 Derivatives at 0: $0, 2, 0, 8, 0, 32$: $\boxed{2x + \dfrac{4}{3}x^3 + \dfrac{4}{15}x^5}$

B Fluency solutions

One mark for each correct part. The notes in italics explain why each sequence was chosen.

CORE

B1 **(a)** $\boxed{5\cosh 5x}$ **(b)** $\boxed{2x\sinh(x^2)}$ **(c)** $\boxed{3\operatorname{sech}^2 3x}$ **(d)** $\boxed{\tanh x}$

Multiply by the inner derivative every time.

B2 **(a)** $\boxed{-\operatorname{sech} x \tanh x}$ **(b)** $\boxed{-2\operatorname{cosech} 2x \coth 2x}$ **(c)** $\boxed{-\operatorname{cosech}^2 x}$ **(d)** $\boxed{-2\operatorname{sech}^2 x \tanh x}$

The reciprocal functions all carry a minus sign.

B3 **(a)** $\boxed{\frac{4}{\sqrt{1-16x^2}}}$ **(b)** $\boxed{-\frac{1}{\sqrt{4-x^2}}}$ **(c)** $\boxed{\frac{2x}{1+x^4}}$ **(d)** $\boxed{\arctan x+\frac{x}{1+x^2}}$

arccos has the minus sign.

B4 **(a)** $\boxed{\frac{3}{\sqrt{9x^2+1}}}$ **(b)** $\boxed{\frac{1}{\sqrt{x^2-4}}}$ **(c)** $\boxed{\frac{3}{1-9x^2}}$ **(d)** $\boxed{\frac{e^x}{\sqrt{e^{2x}+1}}}$

The same results with the chain rule.

B5 **(a)** $\boxed{\frac{4\sqrt{3}}{3}}$ **(b)** $\boxed{\frac{3}{4}}$ **(c)** $\boxed{\frac{1}{4}}$

Substitute after differentiating.

B6 **(a)** $\boxed{\frac{1}{2}}$ **(b)** $\boxed{y-\frac{\pi}{4}=\frac{1}{2}(x-1)}$ **(c)** $\boxed{y-\frac{\pi}{4}=-2(x-1)}$ **(d)** $\boxed{x=1+\frac{\pi}{8}}$

The point is $\left(1,\frac{\pi}{4}\right)$

STRETCH

B7 **(a)** $\frac{dx}{dy}=\cosh y=\sqrt{1+x^2}$: $\boxed{\frac{1}{\sqrt{1+x^2}}}$ **(b)** $\boxed{\frac{1}{\sqrt{x^2+a^2}}}$

(c) $\frac{dx}{dy}=-\operatorname{sech} y\tanh y=-x\sqrt{1-x^2}$: $\boxed{-\frac{1}{x\sqrt{1-x^2}}}$ **(d)** $\boxed{-\frac{25}{12}}$

Write $x=\mathrm{f}(y)$*, then use an identity to return to* x

B8 **(a)** $\boxed{\sinh x-2}$ **(b)** $\boxed{\ln\left(2+\sqrt{5}\right)}$ **(c)** $y''=\cosh x>0$, a minimum **(d)** $\boxed{\sqrt{5}-2\ln\left(2+\sqrt{5}\right)}$

$\cosh x=\sqrt{1+\sinh^2 x}=\sqrt{5}$ *at the point.*

B9 **(a)** $\frac{dy}{dx}=\frac{1}{1+x^2}$ **(b)** differentiate $(1+x^2)y'=1$ **(c)** $(1-x^2)y'=1$, then differentiate **(d)** $y''=9\cosh 3x=9y$

Clear the fraction before differentiating again.

B10 **(a)** $\boxed{x+\frac{x^3}{6}+\frac{x^5}{120}}$ **(b)** $\boxed{1+2x^2+\frac{2}{3}x^4}$ **(c)** $\boxed{x-\frac{x^3}{3}}$ **(d)** $\boxed{\frac{x^2}{2}-\frac{x^4}{12}}$

Like the trigonometric series, without the alternating signs.

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C Reasoning solutions

Your wording can be different. Check that your reason makes the same point.

CORE

C1 The inner derivative 2 is missing: $\boxed{2\sinh 2x}$

C2 The minus sign is missing: $\boxed{-\operatorname{sech} x \tanh x}$

C3 arccos has a minus sign: $\boxed{-\dfrac{3}{\sqrt{1-9x^2}}}$

C4 The chain-rule factor 3 is missing: $\boxed{\dfrac{3}{\sqrt{9x^2-1}}}$

STRETCH

C5 arsech is the inverse of sech, not the reciprocal of arcosh: if $\operatorname{sech} y = \dfrac{x}{2}$ then $\cosh y = \dfrac{2}{x}$:

$$\boxed{\operatorname{arsech}\frac{x}{2} = \operatorname{arcosh}\frac{2}{x}}$$

C6 She did not square-root: $x^2 = \dfrac{1}{2}$: $\boxed{x = \pm\dfrac{1}{\sqrt{2}}}$, both in $(-1, 1)$

C7 $-1 < \tanh x < 1$ for all x, so $\tanh x = -2$ has no solution, and there is no stationary point.

C8 **(a)** $x = \tan y$, $\dfrac{\mathrm{d}x}{\mathrm{d}y} = \sec^2 y = 1 + x^2$, as required

(b) $\dfrac{\mathrm{e}^x}{1+\mathrm{e}^{2x}} = \dfrac{1}{\mathrm{e}^{-x}+\mathrm{e}^x} = \dfrac{1}{2\cosh x} = \dfrac{1}{2}\operatorname{sech} x$, as required

D Problem solving solutions

There is often more than one correct method. Any complete, correct method scores full marks.

CORE

D1 **(a)** $\frac{2}{\sqrt{1-4x^2}} + 2\sqrt{1-4x^2} - \frac{8x^2}{\sqrt{1-4x^2}} = \frac{4-16x^2}{\sqrt{1-4x^2}}$: $\boxed{k = 4}$

(b) $\boxed{2\sqrt{3}}$

D2 **(a)** $\frac{1}{\sqrt{x^2+1}-1} \times \frac{x}{\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}}$ for $x > 0$, as required

(b) both functions have the same derivative and both equal 0 at $x = 0$, so they are equal

D3 $\frac{dy}{dx} = \frac{\cos x}{1-\sin^2 x} = \frac{\cos x}{\cos^2 x} = \sec x$, as required

D4 $\frac{dy}{dx} = \operatorname{arcosh} 2x + \frac{2x}{\sqrt{4x^2-1}}$; at $x = 1$: $\boxed{\ln\left(2+\sqrt{3}\right) + \frac{2\sqrt{3}}{3}}$

STRETCH

D5 **(a)** $\boxed{\frac{2}{1+x^2} - \frac{1}{\sqrt{1+x^2}}}$

(b) $\sqrt{1+x^2} = 2$, $x = \pm\sqrt{3}$: $\boxed{\left(\sqrt{3},\ \frac{2\pi}{3} - \ln\left(2+\sqrt{3}\right)\right),\ \left(-\sqrt{3},\ -\frac{2\pi}{3} + \ln\left(2+\sqrt{3}\right)\right)}$

D6 **(a)** $\boxed{\frac{1}{1-x^2} - \frac{1}{\sqrt{1+x^2}}}$

(b) $\sqrt{1+x^2} = 1-x^2$: $x^4 - 3x^2 = 0$, $x = 0$ or $\pm\sqrt{3}$; $\pm\sqrt{3}$ lie outside $-1 < x < 1$: $\boxed{\textbf{only } x = 0}$

D7 **(a)** $\boxed{1 - \frac{2}{\sqrt{1+x^2}}}$

(b) $x^2 = 3$; the minimum has $x > 0$: $\boxed{P\left(\sqrt{3},\ \sqrt{3} - 2\ln\left(2+\sqrt{3}\right)\right)}$

D8 $n = 1$: $e^x\cosh 2x + 2e^x\sinh 2x = \frac{1}{2}\left(3e^{3x} - e^{-x}\right)$, true. If true for $n = k$, differentiating gives $\frac{1}{2}\left(3^{k+1}e^{3x} + (-1)^{k+1}e^{-x}\right)$, the result for $n = k+1$. True for $n = 1$ and true for $k \Rightarrow$ true for $k+1$, so true for all positive integers n

E Exam practice solutions

Mark-scheme notes are shown on the right of each line.

CORE *Exam questions for everyone*

E1 **(a)** $\frac{1}{1-\frac{x^2}{4}} \times \frac{1}{2} - \frac{1}{x}$ *M1*

$= \frac{2}{4-x^2} - \frac{1}{x}$, as required *A1**

(b) $2x = 4 - x^2$, $x^2 + 2x - 4 = 0$ *M1*

$\boxed{x = \sqrt{5} - 1}$ (the other root is negative) *A1*

(c) $f''(x) = \frac{4x}{(4-x^2)^2} + \frac{1}{x^2} > 0$ *M1*

a minimum *A1*

E2 **(a)** $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}}$ *M1 A1*

$= \frac{1-x}{\sqrt{(1-x)(1+x)}} = \sqrt{\frac{1-x}{1+x}}$, as required *A1**

(b) gradient $\sqrt{\frac{2/5}{8/5}} = \frac{1}{2}$, $y = \arcsin\frac{3}{5} + \frac{4}{5}$ *M1 A1*

$\boxed{y = \frac{1}{2}x + \frac{1}{2} + \arcsin\frac{3}{5}}$ *A1*

(c) $\sqrt{\frac{1-x}{1+x}} > 0$ for $-1 < x < 1$, so $\frac{dy}{dx} \neq 0$ *B1*

STRETCH

E3 **(a)** $y\sqrt{1+x^2} = \operatorname{arsinh} x$; differentiate: $\sqrt{1+x^2}\,y' + \dfrac{xy}{\sqrt{1+x^2}} = \dfrac{1}{\sqrt{1+x^2}}$ *M1 A1*

multiply by $\sqrt{1+x^2}$: $(1+x^2)y' + xy = 1$, as required *A1**

(b) $2xy' + (1+x^2)y'' + y + xy' = 0$ *M1 A1*

$(1+x^2)y'' + 3xy' + y = 0$, as required *A1**

(c) $y(0) = 0$, $y'(0) = 1$, $y''(0) = 0$ *B1*

differentiate again at 0: $y''' + 4y' = 0$, $y'''(0) = -4$ *M1*

$\boxed{y = x - \dfrac{2}{3}x^3}$ *A1*

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