eClassroom

A-LEVEL MATHEMATICS • FURTHER PURE 2

Maclaurin and Taylor Series

Worksheet

Pearson Edexcel International A Level (WFM02) • Calculator allowed: show every step

Free Maclaurin and Taylor Series worksheet for A-Level and International A-Level Maths, with full worked solutions

A **Examples** & Your turn page 2

B **Fluency** core skills page 4

C **Reasoning** explain it page 6

D **Problem** solving page 7

E **Exam** practice page 10

Tap a section to jump straight to it

By the end of this worksheet you can: find Maclaurin series • use the standard series with their validity • combine series • *Stretch:* Taylor series about a point, relations between derivatives, and series solutions of differential equations.

Which questions are for me?

CORE The methods every A-level student must secure. Start here.

STRETCH Series from derivatives and standard results, Taylor series about a point, and series solutions

Every section has both, clearly labelled. Questions get harder as you go, and every skill in Sections A to D is needed in Section E.

Track your progress

Section	B	C	D	E
Core marks	23	10	13	13
Stretch marks	16	14	17	9
My score				

Full worked solutions | Exam questions by topic | Further Pure 2 exam questions | Formula sheet builder

Name ..... Class ..... Date .....

A Worked examples & Your turn

Read each example carefully. Cover it up, then answer the question beside it using the same method.

CORE *Secure the method*

Worked example A1: from derivatives

Find the Maclaurin series of e^{3x} up to x^3

$f^{(n)}(0) = 3^n$; divide by $n!$

$$1 + 3x + \tfrac{9}{2}x^2 + \tfrac{9}{2}x^3$$

Your turn A1

Find the Maclaurin series of $\cos 2x$ up to x^4

Worked example A2: a standard series

Expand $\ln(1+2x)$ up to x^3, and state the validity.

Replace x by $2x$ in $\ln(1+x)$

$$2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3}$$

$$2x - 2x^2 + \tfrac{8}{3}x^3, \ -\tfrac{1}{2} < x \leqslant \tfrac{1}{2}$$

Your turn A2

Expand $(1-3x)^{-1}$ up to x^3, and state the validity.

Worked example A3: multiplying series

Expand $e^x \ln(1+x)$ up to x^3

$$\left(1 + x + \frac{x^2}{2}\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3}\right)$$

$$x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3$$

Your turn A3

Expand $(1+x)e^{-x}$ up to x^3

STRETCH

Worked example A4: a Taylor series

Find the Taylor series of $\ln x$ about $x = 1$ up to $(x-1)^3$

$f(1) = 0, f'(1) = 1, f''(1) = -1, f'''(1) = 2$

$(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$

Your turn A4

Find the Taylor series of $\sqrt{x}$ about $x = 4$ up to $(x-4)^3$

Worked example A5: a series solution

$\frac{dy}{dx} = x + y$, $y(0) = 1$. Find a series for y up to x^3

$y'(0) = 1$, $y'' = 1 + y'$ gives 2, $y''' = y''$ gives 2

$y = 1 + x + x^2 + \frac{1}{3}x^3$

Your turn A5

$\frac{dy}{dx} = xy + 1$, $y(0) = 0$. Find a series for y up to x^3

Worked example A6: a limit

Use series to find $\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$

$\frac{\frac{x^2}{2} + \frac{x^3}{6} + \ldots}{x^2}$

$\frac{1}{2}$

Your turn A6

Use series to find $\lim_{x \to 0} \frac{x - \sin x}{x^3}$

Worked example A7: an approximation

Use $(1+x)^{\frac{1}{2}}$ to estimate $\sqrt{1.02}$

$x = 0.02$: $1 + \frac{x}{2} - \frac{x^2}{8}$

1.00995

Your turn A7

Use the series for $\ln(1+x)$ up to x^3 to estimate $\ln 1.1$

B Fluency

Each question is a designed sequence. From one part to the next, ask yourself: what changed, and what did it do to the answer?

CORE *Secure the method*

B1 Using derivatives, find the Maclaurin series up to the term shown of

(a) e^{-2x}, to x^3

(b) $\sin 3x$, to x^3

(c) $\sqrt{1+x}$, to x^3

(d) $\sec x$, to x^2

[4 marks]

B2 Write down the series, and state when each is valid.

(a) e^{x^2} to x^6

(b) $\cos\frac{x}{2}$ to x^4

(c) $\ln(1-3x)$ to x^3

(d) $(1+2x)^{-\frac{1}{2}}$ to x^2

[4 marks]

B3 Expand up to the term shown

(a) $e^{-x}\cos x$, to x^3

(b) $\ln\frac{1+x}{1-x}$, to x^5

(c) $\cos^2 x$, to x^4

(d) $e^{\sin x}$, to x^3

[4 marks]

B4 Find the Taylor series, up to the square term, of

(a) e^x about $x=2$

(b) $\sin x$ about $x=\frac{\pi}{6}$

(c) $\frac{1}{x}$ about $x=2$

(d) Use (b) to estimate $\sin 0.55$

[4 marks]

B5 $y = xe^x$

(a) Show that $\frac{d^n y}{dx^n} = (x+n)e^x$ for $n = 1, 2, 3$

(b) Find the Maclaurin series up to x^4

(c) Write down the coefficient of x^n for $n \geqslant 1$

[3 marks]

B6 Use series to find

(a) $\lim_{x\to 0}\frac{1-\cos 2x}{x^2}$

(b) $\lim_{x\to 0}\frac{\ln(1+3x)-3x}{x^2}$

(c) $\lim_{x\to 0}\frac{\tan x - x}{x^3}$

(d) $\lim_{x\to 0}\frac{e^{2x}-1-2x}{x^2}$

[4 marks]

STRETCH

B7 $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$, with $y = 1$ and $\frac{dy}{dx} = 0$ at $x = 0$

(a) Find $\frac{d^2y}{dx^2}$ at $x = 0$

(b) Find $\frac{d^3y}{dx^3}$ and its value at $x = 0$

(c) Find $\frac{d^4y}{dx^4}$ at $x = 0$

(d) Write the series up to x^4

[4 marks]

B8 $y = \tan x$, so $\frac{dy}{dx} = 1 + y^2$

(a) Show that $\frac{d^2y}{dx^2} = 2y\frac{dy}{dx}$

(b) Find $\frac{d^3y}{dx^3}$ in terms of y and its derivatives.

(c) Find the values at $x = 0$

(d) Write the series up to x^3

[4 marks]

B9 Expand up to x^2 (or x^3 in (a)), and state the validity.

(a) $\ln(2 + x)$

(b) $\frac{1}{4 - x}$

(c) $\sqrt{9 + x}$

(d) Which of (a), (b), (c) can be used at $x = 3$?

[4 marks]

B10 $\cos x \approx 1 - \frac{x^2}{2} + \frac{x^4}{24}$

(a) Estimate $\cos 0.2$

(b) Find the percentage error.

(c) Estimate $\int_0^{0.5} \cos x \, dx$

(d) Compare with $\sin 0.5$

[4 marks]

C Reasoning

Show your working and explain your thinking. A correct answer without a reason does not score full marks.

CORE *Secure the method*

C1 For $f(x) = e^{3x}$, Amir writes the x^2 term as $9x^2$. Explain his error, and correct it. **[2 marks]**

C2 Bea says the series for $\ln(1+2x)$ is valid for $-1 < x \leqslant 1$. Explain her error, and give the correct range. **[2 marks]**

C3 Given $\frac{dy}{dx} = y^2 + x$, Cal writes $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 + 1$

Explain his error, and find $\frac{d^2y}{dx^2}$ correctly. **[3 marks]**

C4 For $y = e^{2x}\sin x$, Dev writes $y(0) = 1$

Explain his error, and give the first non-zero term of the Maclaurin series. **[3 marks]**

STRETCH

C5 For $y = x^2\cos x$, Eve writes $\frac{d^2y}{dx^2} = 2\cos x - 2x\sin x - x^2\cos x$

Explain her error, and find $\frac{d^2y}{dx^2}$ correctly. **[3 marks]**

C6 Fay writes the Taylor series of $\ln x$ about $x = 1$ as $x - \frac{x^2}{2} + \frac{x^3}{3}$

Explain her error, and correct it. **[3 marks]**

C7 To expand $\frac{\sin x}{1+x}$ up to x^3, Gus uses $(1-x)\left(x - \frac{x^3}{6}\right)$ and obtains $x - x^2 - \frac{x^3}{6}$

Explain his error, and correct the expansion. **[3 marks]**

C8 **(a)** Using the standard Maclaurin series, show that $e^{ix} = \cos x + i\sin x$

(3)

(b) Hence show that $e^{i\pi} + 1 = 0$, and find $e^{\frac{\pi}{2}i}$

(2)

[5 marks]

D Problem solving

You decide the method. Read the whole question first, and show every step.

CORE *Secure the method*

D1 Given that $y = \ln(4+x)$

(a) determine, in simplest form, $\frac{d^3y}{dx^3}$

(1)

(b) Hence determine the Maclaurin series expansion of $\ln(4+x)$, in ascending powers of x up to and including the term in x^3, giving each coefficient in simplest form.

(2)

[3 marks]

D2 The curve C has equation $y = e^{2x}\sin x$

(a) Show that $\frac{d^2y}{dx^2} = 4\frac{dy}{dx} - 5y$

(2)

(b) Hence determine the Maclaurin series expansion of y, in ascending powers of x up to and including the term in x^3, giving each coefficient in simplest form.

(2)

[4 marks]

D3 The variables x and y satisfy the differential equation $\frac{dy}{dx} = xy + y^2$

Given that $y = 1$ at $x = 0$, determine a series solution for y in ascending powers of x, up to and including the term in x^3, giving each coefficient in its simplest form. **[3 marks]**

D4 **(a)** Determine the Taylor series expansion of $\cos x$ about $\frac{\pi}{3}$, in ascending powers of $\left(x - \frac{\pi}{3}\right)$, up to and including the term in $\left(x - \frac{\pi}{3}\right)^2$

(2)

(b) Use your answer to part (a) to determine an approximate value of $\cos 1$, giving your answer to 4 decimal places.

(1)

[3 marks]

STRETCH

D5 *In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.*

(a) Determine, in ascending powers of $\left(x-\frac{\pi}{8}\right)$ up to and including the term in $\left(x-\frac{\pi}{8}\right)^3$, the Taylor series expansion about $\frac{\pi}{8}$ of $y=\tan 2x$, giving each coefficient in simplest form.

(3)

(b) Hence determine an approximate value of $\tan 0.7$, giving your answer to 4 decimal places.

(1)

[4 marks]

D6 Given that $y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=2,\ y>0$

(a) show that $\frac{d^3y}{dx^3}=-\frac{3}{y}\frac{dy}{dx}\frac{d^2y}{dx^2}$

(2)

Given that $y=1$ and $\frac{dy}{dx}=1$ at $x=0$

(b) determine a series solution for y in ascending powers of x, up to and including the term in x^3

(2)

[4 marks]

D7 $y=\ln(1+\sin x),\quad -\frac{\pi}{2}<x<\frac{3\pi}{2}$

(a) Show that $\frac{d^2y}{dx^2}=-\frac{1}{1+\sin x}$

(2)

(b) Hence determine the Maclaurin series expansion of y, in ascending powers of x up to and including the term in x^3

(2)

[4 marks]

D8 The variables x and y satisfy the differential equation

$$\frac{d^2y}{dx^2} = xy + 1$$

Given that $y = 2$ and $\frac{dy}{dx} = 0$ at $x = 1$

(a) determine the values of $\frac{d^3y}{dx^3}$ and $\frac{d^4y}{dx^4}$ at $x = 1$

(3)

(b) Hence determine a series solution for y in ascending powers of $(x-1)$, up to and including the term in $(x-1)^4$

(2)

[5 marks]

E Exam practice

Exam-style questions in the style of WFM02. Write your answers in the spaces, and show every step.

CORE *Exam questions for everyone*

E1 Given that $y = \ln(2+3x)$

(a) determine, in simplest form, $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}$

(2)

(b) Hence determine the Maclaurin series expansion of $\ln(2+3x)$, in ascending powers of x up to and including the term in x^3, giving each coefficient in simplest form.

(2)

(c) Hence write down the Maclaurin series of $\ln(2-3x)$, and use the two series to determine the first two non-zero terms of the Maclaurin series of $\ln\left(\dfrac{2+3x}{2-3x}\right)$

(2)

(Total for Question E1 is 6 marks)

E2 $\dfrac{dy}{dx} = y^2 + 2x$

(a) Show that $\dfrac{d^3y}{dx^3} = 2y\dfrac{d^2y}{dx^2} + 2\left(\dfrac{dy}{dx}\right)^2$

(2)

Given that $y = 1$ at $x = 1$

(b) determine the Taylor series solution for y, in ascending powers of $(x-1)$ up to and including the term in $(x-1)^3$, giving each coefficient in simplest form.

(4)

(c) Use your answer to part (b) to estimate the value of y at $x = 1.1$, giving your answer to 4 decimal places.

(1)

(Total for Question E2 is 7 marks)

STRETCH

E3 Given that $y = e^{-x}\sin 2x$

(a) show that $\dfrac{d^2y}{dx^2} = -2\dfrac{dy}{dx} - 5y$

(3)

(b) Hence, using the result in part (a) to find the higher derivatives, determine the first four non-zero terms of the Maclaurin series expansion of y, giving each coefficient in simplest form.

(5)

(c) Hence find $\lim_{x\to 0}\dfrac{y-2x}{x^2}$

(1)

(Total for Question E3 is 9 marks)

Want more exam practice? The **Exam Questions by Topic** booklets have 10 more full exam questions on Further Pure 2 Chapter 7, Maclaurin and Taylor Series, with a full mark scheme. **Open the booklet**

Finished? Mark your work with the **full worked solutions**, fill in your scores on the front page, then practise real exam questions on the **Further Pure 2 exam questions page**.

★★★★★

Join students already acing their exams.
Book your FREE lesson at **www.eclassroom.org.uk**