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INTERNATIONAL A LEVEL MATHEMATICS • PURE 3 (WMA13)

Pure 3 Practice Paper 1

Worked Solutions

AIMING FOR B

Written in the style of the Pearson Edexcel International Advanced Level paper (WMA13/01) • 75 marks

Every question solved in full on the paper itself, the way your teacher would write it out

How to read these solutions. Each page is the real question page, with a full answer written on the answer lines.

Red pen — the working a full-marks answer shows, with a short note on why each step is taken

M1 A1 — the mark earned on that line: M = method, A = accuracy, B = independent mark

boxed — the final answer to each part

Sit the paper first. These solutions are most useful after you have tried every question yourself under timed conditions. Mark your paper with the mark scheme, then use this booklet to see exactly how each lost mark could have been earned.

Q	Topic	Marks	Page
1	Algebraic fractions and integration	8	2
2	Sign change and iteration	7	3
3	Reciprocal trigonometric equation	7	5
4	Harmonic form in a tide model	8	6
5	Exponential graph and equation	7	8
6	Exponential models and rates	8	9
7	Product rule and a tangent	8	10
8	Differentiating using $\frac{dx}{dy}$	7	12
9	Integration and area	7	13
10	Modulus graph and transformations	8	15

Question paper & mark scheme | Exam questions by topic | More practice papers

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1. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

$$f(x) = \frac{2x}{x-3} - \frac{x+27}{x^2-x-6} \qquad x \in \mathbb{R},\ x \neq -2,\ x \neq 3$$

(a) Show that

$$f(x) = \frac{2x+9}{x+2}$$

(3)

(b) Solve the equation $f(x) = x$

(2)

(c) Find the exact value of

$$\int_4^8 f(x)\,dx$$

giving your answer in the form $a + b\ln c$, where a, b and c are rational constants.

(3)

(a) Factorise the second denominator first: $x^2 - x - 6 = (x-3)(x+2)$

Common denominator: $f(x) = [2x(x+2) - (x+27)] \div [(x-3)(x+2)]$ M1

Numerator: $2x^2 + 4x - x - 27 = 2x^2 + 3x - 27 = (2x+9)(x-3)$ M1

$f(x) = (2x+9)(x-3) \div [(x-3)(x+2)] = \frac{2x+9}{x+2}$ ✓ A1

(b) $\frac{2x+9}{x+2} = x \Rightarrow 2x + 9 = x^2 + 2x \Rightarrow x^2 = 9$ M1

$x = 3$ is not allowed ($x \neq 3$ in the domain of f), so $x = -3$ only A1

(c) Split the fraction: $2x + 9 = 2(x+2) + 5$, so $f(x) = 2 + \frac{5}{x+2}$ M1

$\int \left(2 + \frac{5}{x+2}\right) dx = [\,2x + 5\ln(x+2)\,]$ between $x = 4$ and $x = 8$ M1

$= (16 + 5\ln 10) - (8 + 5\ln 6) = 8 + 5\ln\frac{10}{6}$ $8 + 5\ln(5/3)$ A1

(Total for Question 1 is 8 marks)

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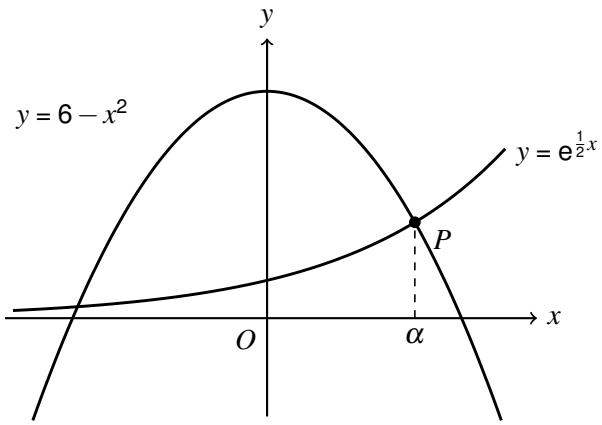

Figure 1

Figure 1 shows part of the curve with equation $y = e^{\frac{1}{2}x}$ and part of the curve with equation $y = 6 - x^2$

The curves meet at the point P, where $x = \alpha$, as shown in Figure 1.

(a) Show that α lies between 1.8 and 1.9

(2)

(b) Show that α is a root of the equation

$$x = \sqrt{6 - e^{\frac{1}{2}x}}$$

(1)

(c) Use the iteration formula

$$x_{n+1} = \sqrt{6 - e^{\frac{1}{2}x_n}}$$

with $x_1 = 1.8$ to find the values of x_2, x_3 and x_4, giving your answers to 4 decimal places.

(2)

A student tries instead to use the iteration formula

$$x_{n+1} = 2\ln\left(6 - x_n^2\right)$$

with $x_1 = 1.8$

(d) Find x_2 and x_3, and explain why this formula cannot be used to find α.

(2)

(a) At P: $e^{x/2} = 6 - x^2$. Let $g(x) = e^{x/2} + x^2 - 6$

$g(1.8) = e^{0.9} + 3.24 - 6 = -0.300\ldots$ $g(1.9) = e^{0.95} + 3.61 - 6 = 0.196\ldots$ M1

Change of sign and g is continuous, so $1.8 < \alpha < 1.9$ ✓ A1

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Question 2 continued

(b) $e^{x/2} = 6 - x^2 \Rightarrow x^2 = 6 - e^{x/2} \Rightarrow x = \sqrt{6 - e^{x/2}}$ ($\alpha > 0$, so the + root) ✓ B1

(c) $x_2 = \sqrt{6 - e^{0.9}} = \sqrt{3.5404\ldots} = 1.88159\ldots$ M1

$x_2 = 1.8816, \quad x_3 = 1.8542, \quad x_4 = 1.8636$ A1

(Keep the full calculator value each time: press = after typing the formula once.)

(d) $x_2 = 2\ln(6 - 1.8^2) = 2\ln 2.76 = 2.0305\ldots$

$x_3 = 2\ln(6 - 2.0305\ldots^2) = 2\ln 1.8772\ldots = 1.2596\ldots$ M1

The values jump either side and move away from $\alpha \approx 1.86$: they diverge A1

so this formula cannot be used to find α.

(Total for Question 2 is 7 marks)

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3. **(a)** Show that the equation

$$2\cot^2\theta + 5\operatorname{cosec}\theta = 10$$

can be written in the form

$$2\operatorname{cosec}^2\theta + 5\operatorname{cosec}\theta - 12 = 0$$

(2)

(b) Hence solve, for $-180^\circ < \theta < 180^\circ$, the equation

$$2\cot^2\theta + 5\operatorname{cosec}\theta = 10$$

giving your answers to one decimal place.

(5)

(a) Use $\cot^2\theta = \operatorname{cosec}^2\theta - 1$: M1

$2(\operatorname{cosec}^2\theta - 1) + 5\operatorname{cosec}\theta = 10 \Rightarrow 2\operatorname{cosec}^2\theta + 5\operatorname{cosec}\theta - 12 = 0$ ✓ A1

(b) A quadratic in $\operatorname{cosec}\theta$: $(2\operatorname{cosec}\theta - 3)(\operatorname{cosec}\theta + 4) = 0$ M1

$\operatorname{cosec}\theta = \frac{3}{2}$ or $\operatorname{cosec}\theta = -4$ A1

$\sin\theta = \frac{1}{\operatorname{cosec}\theta}$: $\sin\theta = \frac{2}{3}$ or $\sin\theta = -\frac{1}{4}$ M1

$\sin\theta = \frac{2}{3}$: $\theta = 41.81^\circ$, $180^\circ - 41.81^\circ = 138.19^\circ$ A1

$\sin\theta = -\frac{1}{4}$: $\theta = -14.48^\circ$, $-180^\circ + 14.48^\circ = -165.52^\circ$ (both in range)

$\theta = -165.5^\circ, -14.5^\circ, 41.8^\circ, 138.2^\circ$ A1

(Total for Question 3 is 7 marks)

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4. The depth of water, D metres, at the entrance to a harbour is modelled by the equation

$$D = 8 + 3\sin\left(\frac{t}{2}\right) - 4\cos\left(\frac{t}{2}\right) \qquad 0 \leqslant t \leqslant 12$$

where t is the number of hours after midnight.

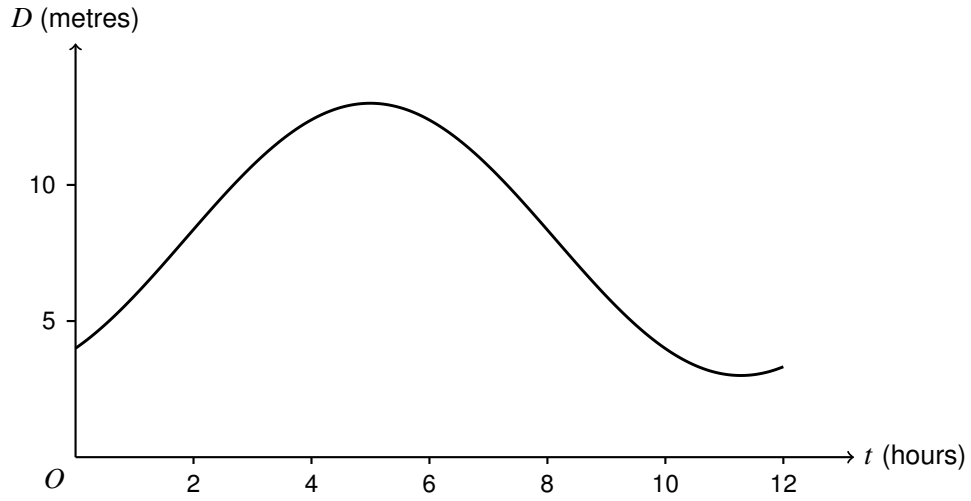

Figure 2

Figure 2 shows the graph of D against t.

(a) Express $3\sin\left(\frac{t}{2}\right) - 4\cos\left(\frac{t}{2}\right)$ in the form $R\sin\left(\frac{t}{2} - \alpha\right)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R, and the value of α in radians to 4 decimal places.

(3)

(b) Find the maximum depth of the water, according to the model, and the value of t at which it occurs.

(2)

(c) Find, according to the model, the times between midnight and midday when the depth of the water is exactly 10 metres. Give each value of t to 2 decimal places.

(3)

(a) $R\sin(\frac{t}{2} - \alpha) = R\sin\frac{t}{2}\cos\alpha - R\cos\frac{t}{2}\sin\alpha$

Compare: $R\cos\alpha = 3$, $R\sin\alpha = 4$. $R = \sqrt{3^2 + 4^2}$ $R = 5$ B1

$\tan\alpha = \frac{4}{3}$ M1

$\alpha = 0.9273$ so $D = 8 + 5\sin(\frac{t}{2} - 0.9273)$ A1

(b) Largest value of sin is 1, when $\frac{t}{2} - \alpha = \frac{\pi}{2} \Rightarrow t = \pi + 2\alpha = 4.996\ldots$ M1

Maximum depth 13 m, when $t = 5.00$ A1

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Question 4 continued

(c) $8 + 5\sin(\frac{t}{2} - \alpha) = 10 \Rightarrow \sin(\frac{t}{2} - \alpha) = 0.4 \Rightarrow \frac{t}{2} - \alpha = 0.4115...$ M1

Second solution: $\frac{t}{2} - \alpha = \pi - 0.4115... = 2.7300...$ (range is -0.93 to 5.07) M1

$t = 2(0.4115... + 0.9273...)$ and $t = 2(2.7300... + 0.9273...)$ $t = 2.68$ and $t = 7.31$ A1

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(Total for Question 4 is 8 marks)

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5. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

(a) Sketch the curve with equation

$$y = 3 - 2e^{-x}$$

On your sketch, show the coordinates of each point where the curve meets the coordinate axes, and state the equation of the asymptote to the curve.

(3)

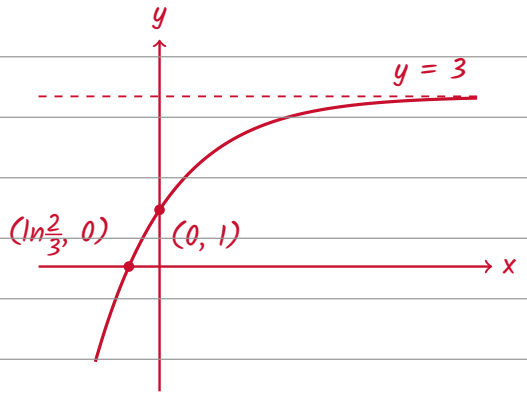
(b) Solve the equation

$$3 - 2e^{-x} = e^{x}$$

giving your answers in exact form.

(4)

(a) As $x \to \infty$, $y \to 3$. $x = 0$: $y = 3 - 2 = 1$. $y = 0$: $e^{-x} = \frac{3}{2}$, $x = \ln\frac{2}{3}$



B1 B1 B1

(b) Multiply every term by e^x ($e^{-x} \times e^x = 1$):

$3e^x - 2 = e^{2x}$ M1

$e^{2x} - 3e^x + 2 = 0$ (a quadratic in e^x) A1

$(e^x - 1)(e^x - 2) = 0 \Rightarrow e^x = 1$ or $e^x = 2$ M1

$x = 0$ or $x = \ln 2$ ($e^x = 1$ gives $x = 0$: do not lose it) A1

(Total for Question 5 is 7 marks)

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6. Two cups of tea, A and B, are made at the same time and left to cool in a room.

The temperature of cup A, $\theta_A\,^\circ$C, t minutes after it is made, is modelled by the equation

$$\theta_A = 20 + 70e^{-0.05t} \qquad t \geqslant 0$$

The temperature of cup B, $\theta_B\,^\circ$C, t minutes after it is made, is modelled by the equation

$$\theta_B = 20 + 60e^{-kt} \qquad t \geqslant 0$$

where k is a positive constant.

Given that the temperature of cup B is 50 °C when $t = 15$

(a) show that $k = \dfrac{\ln 2}{15}$

(2)

(b) Find, according to the models, the value of t when the two cups of tea are at the same temperature.

Give your answer to 3 significant figures.

(3)

(c) Find the rate at which the temperature of each cup is falling at this time, and state which cup is cooling faster.

(3)

(a) $t = 15$, $\theta_B = 50$: $50 = 20 + 60e^{-15k} \Rightarrow e^{-15k} = \frac{30}{60} = \frac{1}{2}$ M1

$-15k = \ln\frac{1}{2} = -\ln 2 \Rightarrow k = \frac{\ln 2}{15}$ ✓ A1

(b) Same temperature: $20 + 70e^{-0.05t} = 20 + 60e^{-kt} \Rightarrow 70e^{-0.05t} = 60e^{-kt}$ M1

$\ln 70 - 0.05t = \ln 60 - kt \Rightarrow t(0.05 - k) = \ln 70 - \ln 60 = \ln\frac{7}{6}$ M1

$t = \frac{0.15415\ldots}{0.05 - 0.046209\ldots} = 40.67\ldots$ $\boxed{t = 40.7 \text{ minutes}}$ A1

(c) $\frac{d\theta_A}{dt} = -3.5e^{-0.05t}$ $\frac{d\theta_B}{dt} = -60ke^{-kt}$ M1

At $t = 40.67\ldots$: $-3.5e^{-2.0335\ldots} = -0.458$ $-60ke^{-1.8794\ldots} = -0.423$ A1

A falls at 0.458 °C per minute, B at 0.423 °C per minute: $\boxed{\text{cup A is cooling faster}}$ A1

(Check: both are 29.2 °C here, and A's rate constant 0.05 is bigger than $k = 0.0462$.)

(Total for Question 6 is 8 marks)

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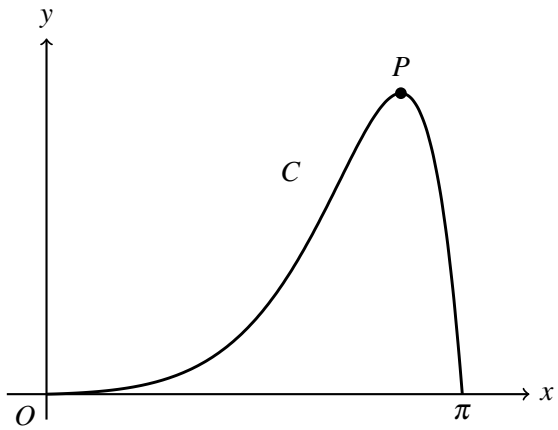

Figure 3

Figure 3 shows a sketch of the curve C with equation

$$y = e^{2x} \sin x \qquad 0 \leqslant x \leqslant \pi$$

The point P is the maximum point on C.

(a) Find $\dfrac{dy}{dx}$

(2)

(b) Show that the x coordinate of P satisfies the equation

$$\tan x = -\frac{1}{2}$$

and hence find the x coordinate of P, giving your answer to 3 significant figures.

(3)

(c) Find the equation of the tangent to C at the point where $x = \dfrac{\pi}{2}$, giving your answer in the form $y = mx + c$, where m and c are exact constants to be found.

(3)

(a) Product rule: $u = e^{2x}$, $u' = 2e^{2x}$; $v = \sin x$, $v' = \cos x$ M1

$dy/dx = 2e^{2x} \sin x + e^{2x} \cos x$ A1

(b) At P: $e^{2x}(2 \sin x + \cos x) = 0$, and $e^{2x} > 0$ always, so $2 \sin x + \cos x = 0$ M1

$2 \sin x = -\cos x \Rightarrow \frac{\sin x}{\cos x} = -\frac{1}{2} \Rightarrow \tan x = -\frac{1}{2}$ ✓ A1

$\arctan(-\frac{1}{2}) = -0.4636\ldots$ is outside 0 to π, so add π: $x = 2.677\ldots$ $x = 2.68$ A1

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Question 7 continued

(c) $x = \frac{\pi}{2}$: $y = e^{\pi} \sin\frac{\pi}{2} = e^{\pi}$, $m = e^{\pi}(2 \times 1 + 0) = 2e^{\pi}$ M1

$y - e^{\pi} = 2e^{\pi}(x - \frac{\pi}{2})$ M1

$y = 2e^{\pi}x - \pi e^{\pi} + e^{\pi}$ $\boxed{y = 2e^{\pi}x + e^{\pi}(1 - \pi)}$ A1

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8. The curve C has equation

$$x = y\mathrm{e}^{y}$$

(a) Find $\dfrac{\mathrm{d}x}{\mathrm{d}y}$

(2)

The point A on C has y coordinate 1

(b) Find an equation of the normal to C at A, giving your answer in the form $y = mx + c$, where m and c are exact constants to be found.

(3)

(c) Find the exact coordinates of the point on C at which the tangent to C is parallel to the y-axis.

(2)

(a) $x = y\,e^y$: product rule with respect to y — M1

$dx/dy = e^y + y\,e^y = e^y(1 + y)$ — A1

(b) $y = 1$: $x = e$, $\frac{dx}{dy} = 2e$, so $\frac{dy}{dx} = \frac{1}{2e}$ (turn it upside down) — M1

Normal gradient $= -2e$: $y - 1 = -2e(x - e)$ — M1

$y = -2ex + 2e^2 + 1$ — A1

(c) Parallel to the y-axis: $\frac{dy}{dx}$ is undefined, so $\frac{dx}{dy} = 0$: $e^y(1 + y) = 0$ — M1

$e^y > 0$, so $y = -1$ and $x = -e^{-1}$ $(-1/e, -1)$ — A1

(Total for Question 8 is 7 marks)

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9. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

(a) Find

$$\int \left(\frac{3}{2x+1} + e^{1-x} \right) dx$$

(2)

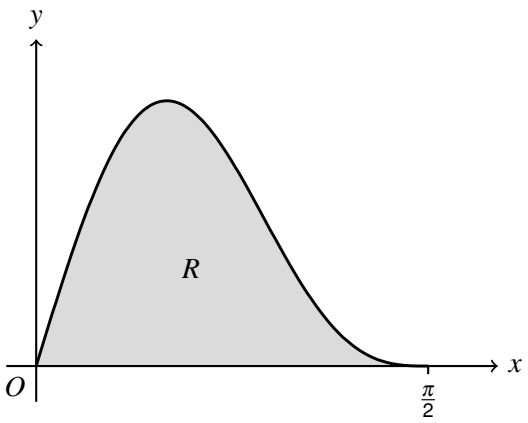

Figure 4

Figure 4 shows part of the curve with equation

$$y = 8 \sin x \cos^3 x \qquad 0 \leqslant x \leqslant \frac{\pi}{2}$$

The region R, shown shaded in Figure 4, is bounded by the curve and the x-axis.

(b) Use calculus to find the exact area of R.

(3)

(c) Show that

$$\int_0^{\frac{\pi}{4}} \tan^2 x \, dx = 1 - \frac{\pi}{4}$$

(2)

(a) $\int \frac{3}{2x+1}\,dx = \frac{3}{2}\ln(2x+1)$ (divide by the 2 inside) B1

$\int e^{1-x}\,dx = -e^{1-x}$ $(3/2)\ln(2x+1) - e^{1-x} + c$ B1

(b) Spot the pattern: $\frac{d}{dx}(\cos^4 x) = 4\cos^3 x \times (-\sin x) = -4 \sin x \cos^3 x$ M1

So $\int 8 \sin x \cos^3 x\,dx = -2\cos^4 x$ A1

Area $= [-2\cos^4 x]$ from 0 to $\frac{\pi}{2}$ $= (-2 \times 0) - (-2 \times 1)$ Area = 2 A1

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Question 9 continued

(c) $\tan^2 x = \sec^2 x - 1$, and $\int \sec^2 x \, dx = \tan x$: $\int (\sec^2 x - 1) \, dx = \tan x - x$ M1

$[\tan x - x]$ from 0 to $\frac{\pi}{4}$ $= (1 - \frac{\pi}{4}) - (0 - 0) = 1 - \frac{\pi}{4}$ ✓ A1

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(Total for Question 9 is 7 marks)

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10. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

The function f is defined by

$$f(x) = 2|x - 3| - 4 \qquad x \in \mathbb{R}$$

(a) Sketch the graph with equation $y = f(x)$, showing the coordinates of the vertex and of each point where the graph meets the coordinate axes.

(3)

(b) Solve the equation

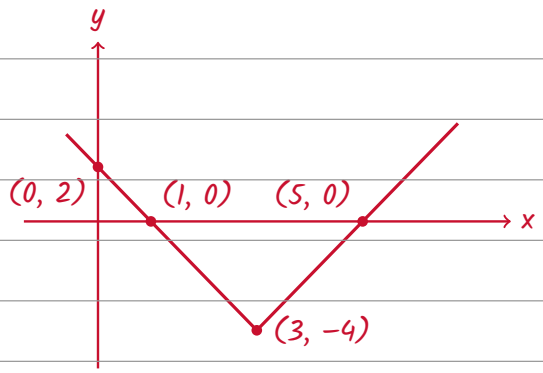
$$f(x) = \frac{1}{2}x$$

(3)

(c) Find the coordinates of the vertex of the graph with equation $y = 3f\left(\frac{1}{2}x\right)$

(2)

(a) Vertex where $x - 3 = 0$: $(3, -4)$. $x = 0$: $2 \times 3 - 4 = 2$. $y = 0$: $|x - 3| = 2$, $x = 1$ or 5



B1 B1 B1

(b) Right branch ($x \geqslant 3$): $2(x - 3) - 4 = \frac{1}{2}x \Rightarrow 2x - 10 = \frac{1}{2}x \Rightarrow x = \frac{20}{3}$ M1

$\frac{20}{3} > 3$ ✓ $x = 20/3$ A1

Left branch ($x < 3$): $-2(x - 3) - 4 = \frac{1}{2}x \Rightarrow 2 - 2x = \frac{1}{2}x \Rightarrow x = \frac{4}{5}$ (< 3 ✓) $x = 4/5$ A1

(c) $f(\frac{1}{2}x)$: stretch by factor 2 parallel to the x-axis, so $x = 3$ becomes $x = 6$ B1

$3f$: stretch by factor 3 parallel to the y-axis, so $y = -4$ becomes $y = -12$ $(6, -12)$ B1

(Total for Question 10 is 8 marks)

TOTAL FOR PAPER IS 75 MARKS

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