Integration

Worked Solutions

Suitable for:

Pearson Edexcel International A-Level (iAL) — Pure Mathematics 2

Question 1 — Worked Solution

Evaluate the definite integral

$$\int_1^5 (3x^2 - 4x + 2)\,\mathrm{d}x = [x^3 - 2x^2 + 2x]_1^5$$

$$= (125 - 50 + 10) - (1 - 2 + 2)$$

$$= 85 - 1$$

$$\therefore = 84$$

Question 2 — Worked Solution

Rewrite as powers, then integrate

$$\int_1^9 \left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)\mathrm{d}x = \left[\frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}}\right]_1^9$$

At x = 9: 2/3 × 27 + 2 × 3 = 18 + 6 = 24

At x = 1: 2/3 + 2 = 8/3

$$= 24 - \frac{8}{3} = \frac{72}{3} - \frac{8}{3}$$

$$\therefore = \frac{64}{3}$$

Question 3 — Worked Solution

Divide each term by √x = x^{1/2}, then integrate

$$\frac{x^2 + 3x}{\sqrt{x}} = \frac{x^2}{x^{\frac{1}{2}}} + \frac{3x}{x^{\frac{1}{2}}} = x^{\frac{3}{2}} + 3x^{\frac{1}{2}}$$

$$\int_1^4 \left(x^{\frac{3}{2}} + 3x^{\frac{1}{2}}\right) dx = \left[\frac{2}{5}x^{\frac{5}{2}} + 2x^{\frac{3}{2}}\right]_1^4$$

At x = 4:   2/5 × 32 + 2 × 8 = 64/5 + 16 = 144/5

At x = 1:   2/5 + 2 = 12/5

$$= \frac{144}{5} - \frac{12}{5}$$

$$\therefore = \frac{132}{5}$$

Question 4 — Worked Solution

(a) Evaluate the definite integral

$$\int_1^3 (x^2 - 4x + 3)\,dx = \left[\frac{x^3}{3} - 2x^2 + 3x\right]_1^3$$

$$= (9 - 18 + 9) - \left(\frac{1}{3} - 2 + 3\right)$$

$$= 0 - \frac{4}{3}$$

$$\therefore = -\frac{4}{3}$$

(b) Area

The integral is negative because the curve lies below the x-axis between x = 1 and x = 3. The area is the absolute value:

$$\therefore \text{Area} = \left| -\frac{4}{3} \right| = \frac{4}{3} \text{ square units}$$

Question 5 — Worked Solution

(a) Evaluate the definite integral

$$\int_0^2 (x^3 - 4x)\,\mathrm{d}x = \left[\frac{x^4}{4} - 2x^2\right]_0^2$$

$$= (4 - 8) - 0$$

$$\therefore = -4$$

(b) Explain and find true area

The answer –4 is negative because the curve lies **below** the x-axis between x = 0 and x = 2 (the curve crosses at x = 0 and x = 2, and is negative between them). The definite integral gives a negative value, not the area.

The area of the shaded region is the absolute value:

$$\therefore \text{Area} = |-4| = 4 \text{ square units}$$

Question 6 — Worked Solution

(a) Intersection points

$$x^2 - 2x = x + 4 \;\Rightarrow\; x^2 - 3x - 4 = 0 \;\Rightarrow\; (x-4)(x+1) = 0$$

$$\therefore x = -1 \quad \text{and} \quad x = 4$$

(b) Area of enclosed region

The line lies above the curve on [–1, 4]:

$$\text{Area} = \int_{-1}^{4} [(x+4) - (x^2 - 2x)]\,\mathrm{d}x = \int_{-1}^{4} (-x^2 + 3x + 4)\,\mathrm{d}x$$

$$= \left[-\frac{x^3}{3} + \frac{3x^2}{2} + 4x\right]_{-1}^{4}$$

$$= \left(-\frac{64}{3} + 24 + 16\right) - \left(\frac{1}{3} + \frac{3}{2} - 4\right)$$

$$= \frac{80}{3} - \left(-\frac{13}{6}\right) = \frac{160}{6} + \frac{13}{6}$$

$$\therefore = \frac{125}{6} \text{ square units}$$

Question 7 — Worked Solution

Total area enclosed by $y = x^3 - 3x$ and the x-axis

By symmetry of an odd function, the two regions have equal area. Compute the integral from $-\sqrt{3}$ to 0 (curve above x-axis):

$$\int_{-\sqrt{3}}^{0}(x^3 - 3x)\,dx = \left[\frac{x^4}{4} - \frac{3x^2}{2}\right]_{-\sqrt{3}}^{0}$$

$$= 0 - \left(\frac{9}{4} - \frac{9}{2}\right) = 0 - \left(-\frac{9}{4}\right) = \frac{9}{4}$$

The integral from 0 to $\sqrt{3}$ gives $-9/4$ (curve is below x-axis). Area of that region = $9/4$.

Total area = $9/4 + 9/4$:

$$\therefore \text{Total area} = \frac{9}{2} \text{ square units}$$

Question 8 — Worked Solution

(a) Find k

Substitute (4, 0) into $y = x^2 + kx$:

$$0 = 16 + 4k \Rightarrow k = -4$$

$$\therefore k = -4$$

(b) x-intercepts

$$y = x^2 - 4x = x(x-4) = 0$$

$$\therefore x = 0 \quad \text{and} \quad x = 4$$

(c) Area enclosed with x-axis

The curve lies below the x-axis between x = 0 and x = 4:

$$\int_0^4 (x^2 - 4x)\,\mathrm{d}x = \left[\frac{x^3}{3} - 2x^2\right]_0^4$$

$$= \left(\frac{64}{3} - 32\right) - 0 = \frac{64}{3} - \frac{96}{3} = -\frac{32}{3}$$

Area = absolute value:

$$\therefore \text{Area} = \frac{32}{3} \text{ square units}$$