eClassroom

A-LEVEL MATHEMATICS • FURTHER PURE 3

Differentiation

Worksheet

Pearson Edexcel International A Level (WFM03) • Calculator allowed: show every step

Free Differentiation worksheet for A-Level and International A-Level Maths, with full worked solutions

A — **Examples** & Your turn — page 2

B — **Fluency** core skills — page 4

C — **Reasoning** explain it — page 6

D — **Problem** solving — page 7

E — **Exam** practice — page 9

Tap a section to jump straight to it

By the end of this worksheet you can: differentiate hyperbolic functions • differentiate inverse trigonometric and hyperbolic functions • *Stretch:* derive these results, find stationary points exactly, and form Maclaurin series.

Which questions are for me?

CORE The methods every A-level student must secure. Start here.

STRETCH Hyperbolic and inverse functions differentiated, derived and used for tangents, stationary points and series.

Every section has both, clearly labelled. Questions get harder as you go, and every skill in Sections A to D is needed in Section E.

Track your progress

Section	B	C	D	E
Core marks	23	10	13	13
Stretch marks	16	14	17	9
My score				

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Name ..... Class ..... Date .....

A Worked examples & Your turn

Read each example carefully. Cover it up, then answer the question beside it using the same method.

CORE *Secure the method*

Worked example A1: the chain rule

Differentiate $y = \cosh^2(2x)$

$2\cosh 2x \times 2\sinh 2x$

$\frac{dy}{dx} = 2\sinh 4x$

Your turn A1

Differentiate $y = \tanh\left(x^2\right)$

Worked example A2: the product rule

Differentiate $y = x^2 \sinh x$

$2x\sinh x + x^2\cosh x$

Your turn A2

Differentiate $y = e^{2x}\cosh x$

Worked example A3: an inverse trigonometric function

Differentiate $y = \arctan 2x$

$\frac{1}{1+(2x)^2} \times 2$

$\frac{2}{1+4x^2}$

Your turn A3

Differentiate $y = \arcsin\frac{x}{3}$

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STRETCH

Worked example A4: an inverse hyperbolic function

Differentiate $y = \operatorname{arcosh} 2x$

$\frac{2}{\sqrt{4x^2-1}}$

Your turn A4

Differentiate $y = \operatorname{artanh} \frac{x}{2}$

Worked example A5: deriving a result

Show that $\frac{\mathrm{d}}{\mathrm{d}x} \operatorname{arcosh} x = \frac{1}{\sqrt{x^2-1}}$

$x = \cosh y$, $\frac{\mathrm{d}x}{\mathrm{d}y} = \sinh y = \sqrt{x^2-1}$ *(for $y > 0$)*

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{x^2-1}}$

Your turn A5

Show that $\frac{\mathrm{d}}{\mathrm{d}x} \arccos x = -\frac{1}{\sqrt{1-x^2}}$

Worked example A6: a stationary point

Find the stationary point of $y = 3\cosh x + \sinh x$

$3\sinh x + \cosh x = 0$, $\tanh x = -\frac{1}{3}$

$x = -\frac{1}{2}\ln 2$

$y = 2\sqrt{2}$

Your turn A6

Find the stationary point of $y = 4\cosh x - 2\sinh x$

Worked example A7: a Maclaurin series

Find the series for $\cosh x$ up to x^4

Derivatives at 0: $1, 0, 1, 0, 1$

$1 + \frac{x^2}{2} + \frac{x^4}{24}$

Your turn A7

Find the series for $\sinh 2x$ up to x^5

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B Fluency

Each question is a designed sequence. From one part to the next, ask yourself: what changed, and what did it do to the answer?

CORE *Secure the method*

B1 Differentiate

(a) $\sinh 5x$

(b) $\cosh\left(x^2\right)$

(c) $\tanh 3x$

(d) $\ln(\cosh x)$

[4 marks]

B2 Differentiate

(a) $\operatorname{sech} x$

(b) $\operatorname{cosech} 2x$

(c) $\coth x$

(d) $\operatorname{sech}^2 x$

[4 marks]

B3 Differentiate

(a) $\arcsin 4x$

(b) $\arccos \frac{x}{2}$

(c) $\arctan\left(x^2\right)$

(d) $x \arctan x$

[4 marks]

B4 Differentiate

(a) $\operatorname{arsinh} 3x$

(b) $\operatorname{arcosh} \frac{x}{2}$

(c) $\operatorname{artanh} 3x$

(d) $\operatorname{arsinh}(e^x)$

[4 marks]

B5 Find the exact gradient of

(a) $y = \arcsin 2x$ at $x = \frac{1}{4}$

(b) $y = \operatorname{arcosh} x$ at $x = \frac{5}{3}$

(c) $y = \arctan x$ at $x = \sqrt{3}$

[3 marks]

B6 The curve $y = \arctan x$

(a) Find the gradient at $x = 1$

(b) Find the tangent at $x = 1$

(c) Find the normal at $x = 1$

(d) Find where the normal meets the x-axis.

[4 marks]

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STRETCH

B7

(a) Derive $\frac{d}{dx}\operatorname{arsinh} x$ from $x = \sinh y$

(b) Hence write down $\frac{d}{dx}\operatorname{arsinh}\frac{x}{a}$

(c) Derive $\frac{d}{dx}\operatorname{arsech} x$ from $x = \operatorname{sech} y$, $0 < x < 1$

(d) Evaluate it at $x = \frac{3}{5}$

[4 marks]

B8 $y = \cosh x - 2x$

(a) Find $\frac{dy}{dx}$

(b) Find the exact x where $\frac{dy}{dx} = 0$

(c) Show that this point is a minimum.

(d) Find the exact minimum value.

[4 marks]

B9 Show that

(a) $y = \arctan x$ gives $(1+x^2)\frac{dy}{dx} = 1$

(b) hence $(1+x^2)y'' + 2xy' = 0$

(c) $y = \operatorname{artanh} x$ gives $(1-x^2)y'' - 2xy' = 0$

(d) $y = \cosh 3x$ gives $y'' = 9y$

[4 marks]

B10 Find the Maclaurin series of

(a) $\sinh x$, up to x^5

(b) $\cosh 2x$, up to x^4

(c) $\tanh x$, up to x^3

(d) $\ln(\cosh x)$, up to x^4

[4 marks]

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C Reasoning

Show your working and explain your thinking. A correct answer without a reason does not score full marks.

CORE *Secure the method*

C1 Amir writes $\frac{\mathrm{d}}{\mathrm{d}x}\cosh 2x = \sinh 2x$. Explain his error, and correct it. **[2 marks]**

C2 Bea writes $\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{sech} x = \operatorname{sech} x \tanh x$. Explain her error, and correct it. **[2 marks]**

C3 Cal writes $\frac{\mathrm{d}}{\mathrm{d}x}\arccos x = \frac{1}{\sqrt{1-x^2}}$

Explain his error, and differentiate $\arccos 3x$ correctly. **[3 marks]**

C4 Dev writes $\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arcosh} 3x = \frac{1}{\sqrt{9x^2-1}}$

Explain his error, and correct it. **[3 marks]**

STRETCH

C5 To differentiate $\operatorname{arsech}\frac{x}{2}$, Eve writes it as $\frac{1}{\operatorname{arcosh}\frac{x}{2}}$

Explain her error, and write $\operatorname{arsech}\frac{x}{2}$ correctly in terms of arcosh. **[3 marks]**

C6 Solving $\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{artanh} x = 2$, Fay obtains $x^2 = \frac{1}{2}$ and writes $x = \frac{1}{2}$

Explain her error, and give the solutions. **[3 marks]**

C7 Gus finds the stationary point of $y = \cosh x + 2\sinh x$ by writing $\tanh x = -2$, so $x = \operatorname{artanh}(-2)$

Explain why the curve has no stationary point. **[3 marks]**

C8 **(a)** Show that $\frac{\mathrm{d}}{\mathrm{d}x}\arctan x = \frac{1}{1+x^2}$

(2)

(b) Hence show that $\frac{\mathrm{d}}{\mathrm{d}x}\arctan(\mathrm{e}^x) = \frac{1}{2}\operatorname{sech} x$

(3)

[5 marks]

D Problem solving

You decide the method. Read the whole question first, and show every step.

CORE *Secure the method*

D1 Given that $y = \arcsin 2x + 2x\sqrt{1-4x^2}$, $\quad |x| < \frac{1}{2}$

(a) show that $\frac{dy}{dx} = k\sqrt{1-4x^2}$, where k is an integer to be determined

(2)

(b) Hence determine the exact value of $\frac{dy}{dx}$ when $x = \frac{1}{4}$

(1)

[3 marks]

D2 Given that $y = \operatorname{arcosh}\sqrt{x^2+1}$, $\quad x > 0$

(a) prove that $\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}}$

(3)

(b) Hence explain why $\operatorname{arcosh}\sqrt{x^2+1} = \operatorname{arsinh} x$ for $x > 0$

(1)

[4 marks]

D3 Given that $y = \operatorname{artanh}(\sin x)$, $\quad -\frac{\pi}{2} < x < \frac{\pi}{2}$, show that $\frac{dy}{dx} = \sec x$ **[3 marks]**

D4 The curve C has equation $y = x\operatorname{arcosh} 2x$, $x \geqslant \frac{1}{2}$

Determine the exact gradient of C at the point where $x = 1$, giving your answer in the form $\ln\left(a+\sqrt{b}\right) + \frac{c\sqrt{3}}{3}$ where a, b and c are integers. **[3 marks]**

STRETCH

D5 $f(x) = 2\arctan x - \operatorname{arsinh} x$

(a) Determine $f'(x)$

(2)

(b) Hence determine the exact coordinates of the stationary points of the curve $y = f(x)$, giving each y-coordinate in terms of π and a natural logarithm.

(2)

[4 marks]

D6 *In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.*

$f(x) = \text{artanh}\, x - \text{arsinh}\, x, \quad -1 < x < 1$

(a) Determine $f'(x)$

(1)

(b) Show that the curve $y = f(x)$ has exactly one stationary point, justifying the rejection of any other values of x that you find.

(3)

[4 marks]

D7 Figure 1 shows a sketch of the curve C with equation $y = x - 2\,\text{arsinh}\, x$

The point P is the minimum turning point of C

(a) Determine $\dfrac{dy}{dx}$

(1)

(b) Determine the exact coordinates of P, giving the y-coordinate in the form $\sqrt{a} - 2\ln\left(b + \sqrt{c}\right)$, where a, b and c are integers.

(3)

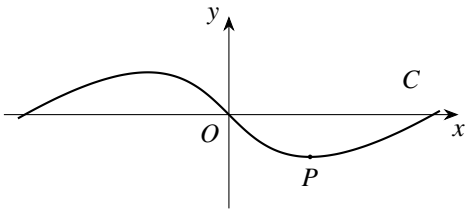


Figure 1

[4 marks]

D8 Given that $y = e^x \cosh 2x$, prove by induction that, for all positive integers n,

$$\frac{d^n y}{dx^n} = \frac{1}{2}\left(3^n e^{3x} + (-1)^n e^{-x}\right)$$

[5 marks]

E Exam practice

Exam-style questions in the style of WFM03. Write your answers in the spaces, and show every step.

CORE *Exam questions for everyone*

E1 The function f is defined by $f(x) = \text{artanh}\left(\frac{x}{2}\right) - \ln x, \quad 0 < x < 2$

(a) Show that $f'(x) = \frac{2}{4-x^2} - \frac{1}{x}$

(2)

(b) Hence determine the exact value of x for which $f'(x) = 0$

(2)

(c) Determine the nature of this stationary point, justifying your answer.

(2)

(Total for Question E1 is 6 marks)

E2 The curve C has equation $y = \arcsin x + \sqrt{1-x^2}, \quad -1 < x < 1$

(a) Show that $\frac{dy}{dx} = \sqrt{\frac{1-x}{1+x}}$

(3)

(b) Determine an equation of the tangent to C at the point where $x = \frac{3}{5}$

(3)

(c) Explain why C has no stationary points.

(1)

(Total for Question E2 is 7 marks)

STRETCH

E3 The variables x and y are related by the equation $y = \dfrac{\text{arsinh}\, x}{\sqrt{1+x^2}}$, where x is real.

(a) Show that $(1+x^2)\dfrac{\mathrm{d}y}{\mathrm{d}x} + xy = 1$

(3)

(b) Hence show that $(1+x^2)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3x\dfrac{\mathrm{d}y}{\mathrm{d}x} + y = 0$

(3)

(c) Hence determine the Maclaurin series expansion of y, in ascending powers of x up to and including the term in x^3

(3)

(Total for Question E3 is 9 marks)

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