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INTERNATIONAL A LEVEL MATHEMATICS • FURTHER PURE MATHEMATICS 3

Further Pure 3

Exam Questions by Topic

Pearson Edexcel International Advanced Level (WFM03/01) • Calculator allowed

Multi-step exam questions at the difficulty of the real WFM03 paper, with a full mark scheme

Chapters (tap a chapter to jump to it)		Qs	Marks	Page	My score
1	**Hyperbolic Functions** Definitions, graphs, inverse functions, identities and equations	10	80	2	
2	**Further Coordinate Systems** Ellipses and hyperbolas, foci and directrices, tangents, normals and loci	10	80	28	
3	**Differentiation** Hyperbolic and inverse trigonometric and hyperbolic functions	10	80	54	
4	**Integration** Substitutions, reduction formulae, arc length and surface area	10	80	80	
5	**Vectors** Vector product, triple products, lines and planes, distances and angles	10	80	106	
6	**Further Matrix Algebra** 3×3 matrices, inverses, eigenvalues and eigenvectors, diagonalisation	10	80	132	

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CHAPTER 1 • KEY SKILLS

1 Hyperbolic Functions

Specification FP3: Hyperbolic functions • 10 questions • 80 marks • about 96 minutes

1 Definitions

$\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right)$,

$\cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right)$, $\tanh x = \dfrac{\sinh x}{\cosh x}$; sech, cosech, coth are reciprocals.

2 Graphs and ranges

$\cosh x \geqslant 1$; $\sinh x$ takes all real values; $-1 < \tanh x < 1$; $0 < \operatorname{sech} x \leqslant 1$

3 Proving identities

When told to use the definitions, write them in full and show a line of working before the result.

4 Osborn's rule

From a trigonometric identity, change the sign of any product of two sines:

$\cosh 2x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$

5 Inverse functions

$\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2 + 1}\right)$,

$\operatorname{arcosh} x = \ln\left(x + \sqrt{x^2 - 1}\right)$ $(x \geqslant 1)$,

$\operatorname{artanh} x = \frac{1}{2}\ln\dfrac{1+x}{1-x}$

6 Exponential equations

For $a\cosh x + b\sinh x = c$, substitute the definitions and multiply by e^{x}: a quadratic in e^{x}

7 Quadratics in cosh or sinh

Use an identity to get one function, solve, then reject $\cosh x < 1$ (and $|\tanh x| \geqslant 1$).

8 Both signs

$\cosh x = k > 1$ gives $x = \pm\ln\left(k + \sqrt{k^2 - 1}\right)$: two answers.

9 Checking roots

$e^{x} > 0$ always. Squaring an equation can introduce extra roots, so substitute back.

Skills map every skill above is tested at least once; interleaved skills come from GCSE or other chapters

Skill	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10
Marks	6	6	6	6	6	10	10	10	10	10
1 Definitions	•	•	•			•	•	•	•	•
2 Graphs and ranges					•					•
3 Proving identities	•	•				•		•	•	•
4 Osborn's rule				•		•		•		
5 Inverse functions		•	•	•	•	•		•	•	•
6 Exponential equations			•				•			
7 Quadratics in cosh or sinh				•		•	•	•	•	
8 Both signs				•		•				•
9 Checking roots			•	•	•	•	•		•	•
Interleaved: exponentials and logarithms (P2, P3)	•	•	•	•		•	•	•	•	•
Interleaved: quadratic equations and surds (P1)		•	•	•		•	•	•	•	•
Interleaved: binomial expansion and $R\cos(x - \alpha)$ (P2, P3)						•				•

Exam difficulty. Questions 1 to 5 check each skill on its own. Questions 6 to 10 are multi-step questions built like the last questions on a real WFM03 paper, where one part leads into the next and skills from different topics are combined.

Must learn. Follow the method the question names: "using the definitions" means exponentials, written in full. Give exact answers as logarithms, and state why any root is rejected.

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1. **(a)** Using the definitions of $\sinh x$ and $\cosh x$ in terms of exponentials, prove that $\cosh^2 x - \sinh^2 x \equiv 1$

(3)

(b) Find the exact values of $\sinh(\ln 3)$, $\cosh(\ln 3)$ and $\tanh(\ln 3)$.

(3)

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Question 1 continued

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2. **(a)** Show that $\operatorname{arcosh} x = \ln\left(x + \sqrt{x^2 - 1}\right)$, $x \geqslant 1$

(4)

(b) Hence write down the exact value of $\operatorname{arcosh} 2$

(1)

(c) Find the exact value of $\operatorname{artanh} \frac{1}{2}$

(1)

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Question 2 continued

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3. Using the definitions of $\cosh x$ and $\sinh x$ in terms of exponentials, solve

$$3\cosh x + \sinh x = 5$$

giving your answers as exact logarithms.

(6)

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Question 3 continued

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4. Solve the equation

$$\cosh 2x - 5\cosh x + 4 = 0$$

giving your answers in exact form.

(6)

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5. **(a)** On the same axes, sketch the graphs of $y = \cosh x$ and $y = \sinh x$.

(2)

(b) Find the range of the function $f(x) = 3 - 2\operatorname{sech} x$, $x \in \mathbb{R}$

(2)

(c) Explain why the equation $\cosh x = \frac{1}{2}$ has no real solutions.

(1)

(d) State the domain and range of $y = \operatorname{arcosh} x$

(1)

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6. **(a)** Using the definition of $\cosh x$ in terms of exponentials, show that

$8\cosh^4 x = \cosh 4x + 4\cosh 2x + 3$

(4)

(b) Hence solve $\cosh 4x + 4\cosh 2x = 15$, giving your answers in the form $\pm \ln k$, where k is an exact surd expression.

(6)

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Question 6 continued

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7. **(a)** Using exponentials, solve $4\tanh x - \operatorname{sech} x = 1$, giving your answer as an exact logarithm.

(6)

(b) A student rearranges to $4\sinh x = 1 + \cosh x$, squares both sides and obtains $\cosh x = \frac{17}{15}$. Explain why this method gives an extra solution, and identify it.

(2)

(c) Verify your answer to part (a) by finding the exact values of $\tanh x$ and $\operatorname{sech} x$.

(2)

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Question 7 continued

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8. **(a)** Show that $\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2 + 1}\right)$

(4)

(b) Solve $2\cosh^2 x - \sinh x = 7$, giving each answer in the form $\ln\left(p + \sqrt{q}\right)$

(4)

(c) Give your answers to part (b) to 3 decimal places.

(2)

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Question 8 continued

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9. **(a)** Starting from the definitions of $\sinh x$ and $\cosh x$ in terms of exponentials, prove that

$$\tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}$$

(4)

(b) Hence solve $\tanh 2x = \frac{4}{5}$, giving your answer as an exact logarithm.

(4)

(c) Check your answer by finding $\operatorname{artanh} \frac{4}{5}$ in logarithmic form.

(2)

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Question 9 continued

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10. **(a)** Express $5\cosh x - 3\sinh x$ in the form $R\cosh(x - \alpha)$, where $R > 0$, giving the exact values of R and α.

(4)

(b) Hence state the minimum value of $5\cosh x - 3\sinh x$ and the exact value of x at which it occurs.

(2)

(c) Solve $5\cosh x - 3\sinh x = 5$, giving your answers in exact form.

(4)

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Question 10 continued

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TOTAL FOR CHAPTER 1 IS 80 MARKS

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CHAPTER 2 • KEY SKILLS

2 Further Coordinate Systems

Specification FP3: Further coordinate systems • 10 questions • 80 marks • about 96 minutes

1 Standard forms

Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $(a\cos\theta, b\sin\theta)$.

Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, $(a\cosh t, b\sinh t)$ or $(a\sec\theta, b\tan\theta)$

2 Eccentricity

Ellipse: $b^2=a^2\left(1-e^2\right)$, $0<e<1$.

Hyperbola: $b^2=a^2\left(e^2-1\right)$, $e>1$.

Take the positive root.

3 Foci and directrices

Foci $(\pm ae, 0)$, given as coordinates; directrices $x=\pm\frac{a}{e}$, given as equations.

4 Focus–directrix property

For any point P on the curve, $PS=e\times PM$, where M is the foot of the perpendicular from P to the directrix.

5 Asymptotes

The hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ has asymptotes $y=\pm\frac{b}{a}x$

6 Tangents

Differentiate parametrically, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta}$, or implicitly, then use $y-y_1=m(x-x_1)$

7 Normals

Gradient $-\frac{1}{m}$; multiply out and collect into the form required, showing each step.

8 Intersections

Substitute one line into the other, keeping the x in equations such as $y=kx$

9 Loci

Find the coordinates of the moving point in terms of the parameter, then eliminate it using $\cos^2\theta+\sin^2\theta=1$ or $\cosh^2 t-\sinh^2 t=1$

Skills map every skill above is tested at least once; interleaved skills come from GCSE or other chapters

Skill	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10
Marks	6	6	6	6	6	10	10	10	10	10
1 Standard forms	•	•	•	•	•	•	•	•	•	•
2 Eccentricity	•	•			•	•			•	•
3 Foci and directrices	•	•				•			•	•
4 Focus–directrix property	•				•				•	•
5 Asymptotes		•					•			
6 Tangents			•			•	•			•
7 Normals				•		•		•		
8 Intersections				•		•	•	•		•
9 Loci					•			•		
Interleaved: parametric and implicit differentiation (P3, P4)			•	•		•	•	•		•
Interleaved: hyperbolic and trigonometric identities (Chapter 1, P3)				•			•	•		
Interleaved: distances, scalar product and areas (P1, P4)	•				•		•		•	•

Exam difficulty. Questions 1 to 5 check each skill on its own. Questions 6 to 10 are multi-step questions built like the last questions on a real WFM03 paper, where one part leads into the next and skills from different topics are combined.

Must learn. Learn which eccentricity formula goes with which curve. Give foci as coordinates and directrices as equations, and show every line when proving a printed tangent or normal.

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1. The ellipse E has equation $\frac{x^2}{25} + \frac{y^2}{16} = 1$

(a) Find the eccentricity of E.

(2)

(b) Write down the coordinates of the foci and the equations of the directrices of E.

(2)

(c) Using the focus–directrix property, show that for any point P on E the sum of the distances from P to the two foci is 10

(2)

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Question 1 continued

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(Total for Question 1 is 6 marks)

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2. The hyperbola H has equation $\frac{x^2}{9} - \frac{y^2}{16} = 1$

(a) Find the eccentricity of H.

(2)

(b) Find the coordinates of the foci and the equations of the directrices of H.

(2)

(c) Write down the equations of the asymptotes of H.

(2)

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Question 2 continued

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3.

(a) Show that the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $(a\cos\theta, b\sin\theta)$ has equation $bx\cos\theta + ay\sin\theta = ab$

(4)

(b) Find the equation of the tangent to $\frac{x^2}{16} + \frac{y^2}{9} = 1$ at the point where $\theta = \frac{\pi}{3}$, giving it in the form $px + qy = r$

(2)

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4. The point $P(4\cosh t, 3\sinh t)$, $t \neq 0$, lies on the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$

(a) Show that the normal to the hyperbola at P has equation $4x\sinh t + 3y\cosh t = 25\sinh t\cosh t$

(5)

(b) Find, in terms of t, the x-coordinate of the point where this normal meets the x-axis.

(1)

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Question 4 continued

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5. The point $P(x, y)$ moves so that its distance from the point $(4, 0)$ is $\frac{2}{3}$ of its distance from the line $x = 9$

(a) Show that the locus of P is the ellipse $\frac{x^2}{36} + \frac{y^2}{20} = 1$

(5)

(b) Verify that the eccentricity of this ellipse is $\frac{2}{3}$

(1)

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Question 5 continued

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6. The ellipse E has equation $\frac{x^2}{9} + \frac{y^2}{4} = 1$

(a) Find the exact eccentricity of E, and hence the coordinates of its foci and the equations of its directrices.

(4)

The point $P(3\cos\theta, 2\sin\theta)$ lies on E, where $0 < \theta < \frac{\pi}{2}$

(b) Show that the tangent to E at P has equation $2x\cos\theta + 3y\sin\theta = 6$

(3)

(c) The line l passes through the origin and is perpendicular to this tangent. Find an equation of l, and the x-coordinate of the point where l meets the tangent.

(3)

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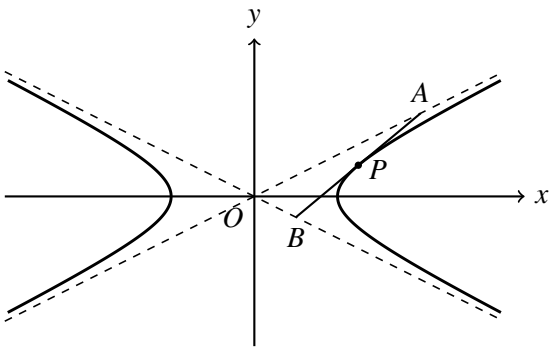

Figure 1

The hyperbola H has equation $\frac{x^2}{4} - y^2 = 1$. The point $P(2\sec\theta, \tan\theta)$, $0 < \theta < \frac{\pi}{2}$, lies on H. The tangent to H at P meets the asymptotes of H at A and B, as shown in Figure 1.

(a) Show that the tangent at P has equation $x\sec\theta - 2y\tan\theta = 2$

(4)

(b) Show that P is the midpoint of AB.

(4)

(c) Show that the area of triangle OAB is the same for every position of P, and state its value.

(2)

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Question 7 continued

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8. The point $P(4\cos\theta, 3\sin\theta)$, $0 < \theta < \frac{\pi}{2}$, lies on the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$

(a) Show that the normal to the ellipse at P has equation $\frac{4x}{\cos\theta} - \frac{3y}{\sin\theta} = 7$

(4)

The normal meets the x-axis at A and the y-axis at B, and M is the midpoint of AB.

(b) Find the coordinates of A and B in terms of θ.

(2)

(c) Show that, as θ varies, M lies on the curve $64x^2 + 36y^2 = 49$

(4)

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Question 8 continued

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(Total for Question 8 is 10 marks)

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9. The hyperbola H has equation $\frac{x^2}{16} - \frac{y^2}{9} = 1$

(a) Find the eccentricity of H, the coordinates of its foci S and S', where S has positive x-coordinate, and the equations of its directrices.

(3)

(b) The point P on H has x-coordinate 8 and positive y-coordinate. Using the focus–directrix property, find PS, and check your answer by finding the coordinates of P.

(3)

(c) Show that, for any point P on the branch of H with $x > 0$, $PS' - PS = 8$

(4)

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10.

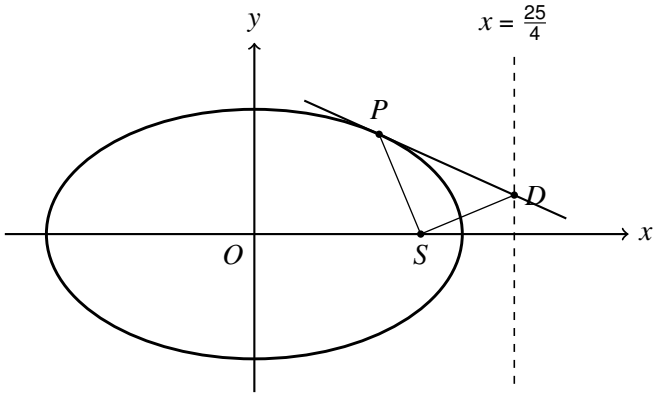

Figure 2

The ellipse E has equation $9x^2 + 25y^2 = 225$. The point $P(5\cos\theta, 3\sin\theta)$, $0 < \theta < \frac{\pi}{2}$, lies on E.

(a) Find the eccentricity of E, and write down the coordinates of the focus S with positive x-coordinate and the equation of the corresponding directrix.

(3)

(b) Show that the tangent to E at P has equation $3x\cos\theta + 5y\sin\theta = 15$

(3)

(c) The tangent meets the x-axis at T, and N is the foot of the perpendicular from P to the x-axis. Show that $OT \times ON = 25$

(2)

(d) The tangent meets the directrix at D, as shown in Figure 2. Show that angle PSD is a right angle.

(2)

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TOTAL FOR CHAPTER 2 IS 80 MARKS

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CHAPTER 3 • KEY SKILLS

3 Differentiation

Specification FP3: Differentiation • 10 questions • 80 marks • about 96 minutes

1 Hyperbolic derivatives

$\sinh \to \cosh$, $\cosh \to \sinh$,

$\tanh \to \operatorname{sech}^2$, $\operatorname{sech} \to -\operatorname{sech}\tanh$,

$\operatorname{cosech} \to -\operatorname{cosech}\coth$,

$\coth \to -\operatorname{cosech}^2$

2 Chain, product, quotient

$\frac{\mathrm{d}}{\mathrm{d}x}\tanh^2(2x) = 2\tanh 2x \times 2\operatorname{sech}^2 2x$:
do not lose the inner derivative.

3 Inverse trigonometric

$\arcsin x \to \frac{1}{\sqrt{1-x^2}}$,

$\arccos x \to -\frac{1}{\sqrt{1-x^2}}$,

$\arctan x \to \frac{1}{1+x^2}$

4 Inverse hyperbolic

$\operatorname{arsinh} x \to \frac{1}{\sqrt{x^2+1}}$,

$\operatorname{arcosh} x \to \frac{1}{\sqrt{x^2-1}}$,

$\operatorname{artanh} x \to \frac{1}{1-x^2}$

5 Deriving them

Write $x = \mathrm{f}(y)$, find $\frac{\mathrm{d}x}{\mathrm{d}y}$, use an identity to write it in x, then take the reciprocal.

6 Stationary points

Set $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$; equations such as $2\sinh x + \cosh x = 0$ give $\tanh x = -\frac{1}{2}$

7 Exact values

Use $\cosh x = \frac{1}{\sqrt{1-\tanh^2 x}}$ or logarithms to give exact coordinates.

8 Differential equations

Find y' and y'', then substitute; clearing a square root first often helps.

9 Valid solutions

Check domains: $|x| < 1$ for artanh, $x \geqslant 1$ for arcosh; reject values outside them.

Skills map every skill above is tested at least once; interleaved skills come from GCSE or other chapters

Skill	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10
Marks	6	6	6	6	6	10	10	10	10	10
1 Hyperbolic derivatives	•			•		•	•			
2 Chain, product, quotient	•	•	•			•	•	•	•	
3 Inverse trigonometric		•			•			•		
4 Inverse hyperbolic			•			•			•	•
5 Deriving them		•	•			•				
6 Stationary points				•			•			•
7 Exact values				•		•	•			•
8 Differential equations					•				•	
9 Valid solutions						•		•		•
Interleaved: hyperbolic identities and logarithms (Chapter 1)			•	•		•	•			•
Interleaved: tangents, stationary points, second derivative (P2, P3)				•		•	•			•
Interleaved: trigonometric identities (P3)		•			•			•		

Exam difficulty. Questions 1 to 5 check each skill on its own. Questions 6 to 10 are multi-step questions built like the last questions on a real WFM03 paper, where one part leads into the next and skills from different topics are combined.

Must learn. Write every "h" clearly. When deriving an inverse derivative, show $x = \mathrm{f}(y)$, the derivative in y, the identity used and the final form in x.

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1. Differentiate with respect to x

(a) $\cosh 3x \sinh 2x$

(2)

(b) $\tanh^2(2x)$

(2)

(c) $\ln(\cosh x)$

(1)

(d) $\operatorname{sech} x$

(1)

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Question 1 continued

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2. **(a)** Given that $y = \arcsin x$, $-1 < x < 1$, show that $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}}$

(3)

(b) Find $\dfrac{d}{dx}(\arctan 3x)$

(1)

(c) Find $\dfrac{d}{dx}\left(\arccos x^2\right)$, $0 < x < 1$

(2)

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Question 2 continued

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3. **(a)** Given that $y = \text{artanh}\, x$, $|x| < 1$, show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{1-x^2}$

(3)

(b) Find $\dfrac{\mathrm{d}}{\mathrm{d}x}(\text{arsinh}\, 2x)$

(1)

(c) Find $\dfrac{\mathrm{d}}{\mathrm{d}x}(x\,\text{arcosh}\, x)$, $x > 1$

(2)

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Question 3 continued

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4. The curve C has equation $y = 2\cosh x + \sinh x$

(a) Find the exact coordinates of the stationary point of C.

(4)

(b) Determine the nature of the stationary point.

(2)

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Question 4 continued

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5. The function $y = \arcsin x$ is defined for $-1 < x < 1$. By finding its first and second derivatives, prove that y is a solution of the differential equation

$$\left(1 - x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = x\frac{\mathrm{d}y}{\mathrm{d}x}$$

(6)

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Question 5 continued

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6. **(a)** Given that $y = \text{arsech}\left(\frac{x}{2}\right)$, $0 < x < 2$, show that $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2}{x\sqrt{4-x^2}}$

(5)

(b) Find the exact value of x, $0 < x < 1$, at which the gradient of $y = \text{artanh}\, x$ is $\frac{4}{3}$, explaining why any other value is rejected.

(2)

(c) Find an equation of the tangent to $y = \text{arsech}\left(\frac{x}{2}\right)$ at the point where $x = 1$, giving the y-intercept as an exact expression involving a logarithm.

(3)

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Question 6 continued

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7. The curve C has equation $y = \mathrm{e}^{-x}\cosh 2x$

(a) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{-x}(2\sinh 2x - \cosh 2x)$

(2)

(b) Show that the x-coordinate of the stationary point of C is $\frac{1}{4}\ln 3$

(3)

(c) Find the exact y-coordinate of the stationary point, in the form $k \times 3^n$

(3)

(d) Determine the nature of the stationary point.

(2)

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Question 7 continued

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8. **(a)** By differentiating, show that $\arctan x + \arctan \frac{1}{x}$ is constant for $x > 0$, and find the constant.

(4)

(b) Show that, for $|x| < 1$, $\frac{\mathrm{d}}{\mathrm{d}x} \arcsin\left(\frac{2x}{1+x^2}\right) = \frac{2}{1+x^2}$

(4)

(c) Hence show that $\arcsin\left(\frac{2x}{1+x^2}\right) = 2\arctan x$ for $|x| < 1$

(2)

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Question 8 continued

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Question 8 continued

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9.

(a) Given that $y = \text{arsinh}\, x$, show that $\left(1+x^2\right)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+x\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$

(5)

(b) Given instead that $y = (\text{arsinh}\, x)^2$, show that $\left(1+x^2\right)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+x\dfrac{\mathrm{d}y}{\mathrm{d}x}=2$

(5)

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Question 9 continued

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10.

$$f(x) = 2\operatorname{arcosh} x - x, \qquad x \geqslant 1$$

(a) Find the exact coordinates of the stationary point of $y = f(x)$, and show that it is a maximum.

(6)

(b) Find an equation of the tangent to $y = f(x)$ at the point where $x = \frac{5}{3}$, giving exact values.

(4)

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Question 10 continued

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TOTAL FOR CHAPTER 3 IS 80 MARKS

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CHAPTER 4 • KEY SKILLS

4 Integration

Specification FP3: Integration • 10 questions • 80 marks • about 96 minutes

1 Hyperbolic integrals

$\sinh x \to \cosh x$, $\cosh x \to \sinh x$, $\tanh x \to \ln\cosh x$; use $\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$

2 Standard results

$\frac{1}{\sqrt{a^2+x^2}} \to \operatorname{arsinh}\frac{x}{a}$;

$\frac{1}{\sqrt{x^2-a^2}} \to \operatorname{arcosh}\frac{x}{a}$;

$\frac{1}{a^2+x^2} \to \frac{1}{a}\arctan\frac{x}{a}$;

$\frac{1}{\sqrt{a^2-x^2}} \to \arcsin\frac{x}{a}$

3 Completing the square

$x^2+4x+13 = (x+2)^2+9$; with $-x^2$, write $a^2-(x-p)^2$. Never take a minus sign out of a square root.

4 Coefficient of x^2

$\sqrt{64-(2x-1)^2}$: with $u = 2x-1$, $\mathrm{d}x = \frac{1}{2}\mathrm{d}u$, so keep the factor $\frac{1}{2}$

5 Substitutions

$x = a\sinh u$ for $\sqrt{a^2+x^2}$, $x = a\cosh u$ for $\sqrt{x^2-a^2}$, $x = a\sin u$ for $\sqrt{a^2-x^2}$; change the limits.

6 Reduction formulae

Integrate by parts to relate I_n to I_{n-1} or I_{n-2}; bracket carefully, then work down to I_0 or I_1

7 Arc length

$s = \int \sqrt{1+y'^2}\,\mathrm{d}x$ or $\int \sqrt{\dot{x}^2+\dot{y}^2}\,\mathrm{d}t$.

Simplify the derivative before squaring.

8 Surface area

Rotating about the x-axis:

$S = 2\pi\int y\,\mathrm{d}s$. "Total" surface area adds any end discs.

9 Exact answers

Give logs and surds exactly: $\operatorname{arsinh}1 = \ln\left(1+\sqrt{2}\right)$; write the h in arsinh, arcosh, artanh.

Skills map every skill above is tested at least once; interleaved skills come from GCSE or other chapters

Skill	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10
Marks	6	6	6	6	6	10	10	10	10	10
1 Hyperbolic integrals	•			•	•			•		•
2 Standard results		•	•				•		•	
3 Completing the square			•						•	
4 Coefficient of x^2									•	
5 Substitutions				•			•	•		
6 Reduction formulae						•				
7 Arc length					•		•			•
8 Surface area							•			•
9 Exact answers	•	•	•	•	•	•	•	•	•	•
Interleaved: integration by parts and substitution (P4)				•		•	•	•		
Interleaved: hyperbolic functions and identities (Chapter 1)	•	•		•	•		•	•		•
Interleaved: parametric differentiation and volumes (P4)							•			•

Exam difficulty. Questions 1 to 5 check each skill on its own. Questions 6 to 10 are multi-step questions built like the last questions on a real WFM03 paper, where one part leads into the next and skills from different topics are combined.

Must learn. Complete the square before reaching for the formula book, change the limits with every substitution, and read whether the question asks for a curved or a total surface area.

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1.

(a) Find $\int \cosh^2 x \, \mathrm{d}x$

(2)

(b) Find the exact value of $\int_0^{\ln 2} \sinh x \, \mathrm{d}x$

(2)

(c) Find $\int \tanh x \, \mathrm{d}x$

(2)

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Question 1 continued

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2. Using standard integrals from the formula book, evaluate exactly

(a) $\int_0^3 \frac{1}{\sqrt{9+x^2}}\,\mathrm{d}x$, as a logarithm,

(2)

(b) $\int_0^2 \frac{1}{4+x^2}\,\mathrm{d}x$

(2)

(c) $\int_3^5 \frac{1}{\sqrt{x^2-9}}\,\mathrm{d}x$, giving your answer as a logarithm.

(2)

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Question 2 continued

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3. By completing the square, find

(a) $\int \frac{1}{\sqrt{x^2+4x+13}}\,\mathrm{d}x$

(3)

(b) $\int \frac{1}{\sqrt{8+2x-x^2}}\,\mathrm{d}x$

(3)

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Question 3 continued

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4. Use the substitution $x = 2\sinh u$ to find the exact value of

$$\int_0^2 \sqrt{4+x^2}\,\mathrm{d}x$$

giving your answer in the form $a\ln\left(1+\sqrt{2}\right)+b\sqrt{2}$

(6)

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Question 4 continued

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5. **(a)** Find the length of the arc of the curve $y = \cosh x$ from $x = 0$ to $x = \ln 3$

(3)

(b) Find the length of the arc of the curve $y = \frac{2}{3}x^{\frac{3}{2}}$ from $x = 0$ to $x = 3$

(3)

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Question 5 continued

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(Total for Question 5 is 6 marks)

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6.

$$I_n = \int_0^{\frac{\pi}{2}} x^n \sin x \, \mathrm{d}x, \qquad n \geqslant 0$$

(a) Show that, for $n \geqslant 2$, $I_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}$

(5)

(b) Hence find the exact value of I_4, giving your answer in the form $a\pi^3 + b\pi + c$

(5)

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Question 6 continued

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7. The curve C has parametric equations $x = t^2$, $y = 2t$, $0 \leqslant t \leqslant 1$

(a) Show that the length of C is $\sqrt{2} + \ln\left(1 + \sqrt{2}\right)$

(5)

(b) A bowl is modelled by spinning C completely about the x-axis. Calculate the exact area of the bowl's curved surface.

(5)

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Question 7 continued

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(Total for Question 7 is 10 marks)

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8.

$$I = \int_4^5 \sqrt{x^2 - 16}\,\mathrm{d}x$$

(a) Use the substitution $x = 4\cosh\theta$ to show that $I = 16\int_0^{\ln 2} \sinh^2\theta\,\mathrm{d}\theta$

(3)

(b) Hence show that $I = \frac{15}{2} - 8\ln 2$

(5)

(c) Interpret I as an area, and check that its value is reasonable.

(2)

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Question 8 continued

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9.

(a) Find $\int \frac{1}{\sqrt{63+4x-4x^2}}\,\mathrm{d}x$

(5)

(b) Find the exact value of $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{4x^2+4x+5}\,\mathrm{d}x$

(5)

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Question 9 continued

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10. The curve C has equation $y = 2\cosh\left(\frac{x}{2}\right)$, $0 \leqslant x \leqslant 2\ln 3$

(a) Find the length of C.

(3)

(b) The curve is rotated through 2π radians about the x-axis. Show that the area of the curved surface generated is $4\pi\ln 3 + \frac{80\pi}{9}$

(5)

(c) Find the exact **total** surface area of the solid formed.

(2)

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Question 10 continued

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TOTAL FOR CHAPTER 4 IS 80 MARKS

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CHAPTER 5 • KEY SKILLS

5 Vectors

Specification FP3: Vectors • 10 questions • 80 marks • about 96 minutes

1 Vector product

$\mathbf{a} \times \mathbf{b}$ by the determinant method; it is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, and $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$

2 Areas

Triangle ABC: $\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$;

parallelogram: $\left|\overrightarrow{AB} \times \overrightarrow{AD}\right|$

3 Volumes

Scalar triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$:

parallelepiped $|\ldots|$, tetrahedron $\frac{1}{6}|\ldots|$

4 Lines

$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, or $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$, i.e. $\mathbf{r} \times \mathbf{b} = \mathbf{a} \times \mathbf{b}$, or Cartesian $\dfrac{x-a_1}{b_1} = \dfrac{y-a_2}{b_2} = \dfrac{z-a_3}{b_3}$

5 Planes

$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$, i.e. $n_1x + n_2y + n_3z = d$; from $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, take $\mathbf{n} = \mathbf{b} \times \mathbf{c}$

6 Intersections

Line and plane: substitute the line into the plane equation. Two planes: direction $\mathbf{n}_1 \times \mathbf{n}_2$, plus one common point.

7 Angles

Between a line and a plane: $\sin\theta = \dfrac{|\mathbf{b} \cdot \mathbf{n}|}{|\mathbf{b}||\mathbf{n}|}$. Between planes: the angle between the normals.

8 Distances

Point to plane: $\dfrac{|n_1x_0 + n_2y_0 + n_3z_0 - d|}{|\mathbf{n}|}$. Skew lines: $\dfrac{|(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{b}_1 \times \mathbf{b}_2)|}{|\mathbf{b}_1 \times \mathbf{b}_2|}$

9 Feet and reflections

Move from the point along $\mathbf{n}$ to the plane (foot), then the same distance again (reflection). Answer lines as "$\mathbf{r} = \ldots$".

Skills map every skill above is tested at least once; interleaved skills come from GCSE or other chapters

Skill	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10
Marks	6	6	6	6	6	10	10	10	10	10
1 Vector product	•	•	•				•	•	•	•
2 Areas	•								•	
3 Volumes		•							•	
4 Lines				•		•	•	•		•
5 Planes			•	•	•	•		•	•	•
6 Intersections				•		•	•	•		
7 Angles				•			•	•		
8 Distances					•	•	•		•	•
9 Feet and reflections					•	•				•
Interleaved: vectors and scalar product (P4)	•	•	•	•	•	•	•	•	•	•
Interleaved: simultaneous equations (P1)							•	•		
Interleaved: trigonometry and surds (P1)	•			•			•	•	•	

Exam difficulty. Questions 1 to 5 check each skill on its own. Questions 6 to 10 are multi-step questions built like the last questions on a real WFM03 paper, where one part leads into the next and skills from different topics are combined.

Must learn. Sketch the situation, choose the right vectors (direction and normal), and say which angle you have found before adjusting it. Write lines as "$\mathbf{r} =$" and planes with their constant.

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1. The vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} + 2\mathbf{k}$ are the position vectors of the points A and B.

(a) Find $\mathbf{a} \times \mathbf{b}$.

(2)

(b) Verify that $\mathbf{a} \times \mathbf{b}$ is perpendicular to $\mathbf{a}$.

(1)

(c) Find the exact area of triangle OAB.

(2)

(d) Find a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

(1)

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Question 1 continued

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(Total for Question 1 is 6 marks)

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2. The points A, B and C have position vectors $\mathbf{i}+2\mathbf{k}$, $2\mathbf{i}+\mathbf{j}$ and $3\mathbf{j}+\mathbf{k}$ respectively, and O is the origin.

(a) Find the volume of the tetrahedron $OABC$.

(4)

(b) Explain what it would mean if the scalar triple product $\overrightarrow{OA}\cdot\left(\overrightarrow{OB}\times\overrightarrow{OC}\right)$ were zero.

(2)

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Question 2 continued

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3. The plane Π passes through the points $A(1,0,2)$, $B(3,1,1)$ and $C(2,-1,4)$.

(a) Find a Cartesian equation of Π.

(4)

(b) Verify that C lies on your plane.

(1)

(c) Write the equation of Π in the form $\mathbf{r} \cdot \mathbf{n} = p$

(1)

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Question 3 continued

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(Total for Question 3 is 6 marks)

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4. The line l has equation $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \lambda(2\mathbf{i} - \mathbf{j} + 3\mathbf{k})$ and the plane Π has equation $x + 2y + 2z = 13$

(a) Find the coordinates of the point where l meets Π.

(3)

(b) Find the acute angle between l and Π, to the nearest 0.1°

(3)

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Question 4 continued

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(Total for Question 4 is 6 marks)

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5. The point P has coordinates $(3, -1, 2)$ and the plane Π has equation $2x - y + 2z = 1$

(a) Find the perpendicular distance from P to Π.

(3)

(b) Find the coordinates of the foot of the perpendicular from P to Π.

(3)

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Question 5 continued

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(Total for Question 5 is 6 marks)

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6. The line l passes through $A(2, 1, -1)$ in the direction $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$. The plane Π has equation $2x - 2y + z = 9$

(a) Show that l is parallel to Π, and find the distance between them.

(5)

(b) Find the coordinates of the reflection A' of A in Π.

(3)

(c) Write down a vector equation of the reflection of l in Π.

(2)

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Question 6 continued

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7. The lines l_1 and l_2 have equations

$$l_1 : \mathbf{r} = \mathbf{i} + 2\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j}), \qquad l_2 : \mathbf{r} = 3\mathbf{j} + \mathbf{k} + \mu(2\mathbf{i} + \mathbf{k})$$

(a) Find the shortest distance between l_1 and l_2, giving an exact answer.

(5)

(b) Explain why your answer shows that the lines do not intersect.

(2)

(c) Find the acute angle between the directions of l_1 and l_2, to the nearest 0.1°

(3)

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8. The planes Π_1 and Π_2 have equations $x+y+z=6$ and $2x-y+z=3$ respectively.

(a) Find the acute angle between Π_1 and Π_2, to the nearest 0.1°

(3)

(b) Find a vector equation of the line of intersection of Π_1 and Π_2.

(4)

(c) Write this line in Cartesian form.

(1)

(d) Write this line in the form $\mathbf{r}\times\mathbf{b}=\mathbf{c}$

(2)

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Question 8 continued

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9. A tetrahedron has vertices $A(1,1,0)$, $B(3,0,1)$, $C(2,3,1)$ and $D(1,2,4)$.

(a) Find the exact area of the face ABC.

(3)

(b) Find the volume of the tetrahedron.

(3)

(c) Hence find the exact perpendicular distance from D to the plane ABC.

(2)

(d) Find a Cartesian equation of the plane ABC, and use it to check your answer to part (c).

(2)

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Question 9 continued

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10. The plane Π has vector equation $\mathbf{r} = \mathbf{i} + s(\mathbf{i} + \mathbf{j}) + t(\mathbf{j} + \mathbf{k})$

(a) Find a Cartesian equation of Π.

(3)

(b) The line l has equation $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} + \lambda(\mathbf{i} - \mathbf{k})$. Show that l is parallel to Π and find the exact distance between them.

(3)

(c) Find the coordinates of the point of Π nearest to the origin.

(2)

(d) Find a Cartesian equation of the plane that contains l and is perpendicular to Π.

(2)

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TOTAL FOR CHAPTER 5 IS 80 MARKS

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CHAPTER 6 • KEY SKILLS

6 Further Matrix Algebra

Specification FP3: Further matrix algebra • 10 questions • 80 marks • about 96 minutes

1 3×3 determinants

Expand along a row or column with zeros. Singular means det = 0; $\det(\mathbf{AB}) = \det\mathbf{A}\det\mathbf{B}$

2 Inverse

Minors, then cofactors (sign pattern), transpose to get the adjugate, then divide by the determinant.

3 Products

$(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$ and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$: the order reverses.

4 Linear equations

Write $\mathbf{Mx} = \mathbf{c}$; if $\det\mathbf{M} \neq 0$ the three planes meet at the single point $\mathbf{x} = \mathbf{M}^{-1}\mathbf{c}$

5 Transformations

The image of $\mathbf{a} + t\mathbf{b}$ is $\mathbf{Ma} + t\mathbf{Mb}$; the point with image $\mathbf{c}$ is $\mathbf{M}^{-1}\mathbf{c}$

6 Eigenvalues

Solve $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$, not $\det\mathbf{M} = 0$. If one eigenvalue is given, substitute it straight away.

7 Eigenvectors

Solve $(\mathbf{M} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ (equivalently $\mathbf{Mx} = \lambda\mathbf{x}$). Normalise by dividing by the magnitude.

8 Diagonal form

For symmetric $\mathbf{A}$, the columns of $\mathbf{P}$ are normalised, orthogonal eigenvectors; $\mathbf{P}^{\mathrm{T}}\mathbf{AP} = \mathbf{D}$ with eigenvalues in the same order.

9 Repeated eigenvalues

Choose two eigenvectors for the repeated value that are perpendicular to each other (check the scalar product is 0).

Skills map every skill above is tested at least once; interleaved skills come from GCSE or other chapters

Skill	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10
Marks	6	6	6	6	6	10	10	10	10	10
1 3×3 determinants	•	•			•	•	•	•		•
2 Inverse		•	•				•	•		
3 Products	•		•							
4 Linear equations							•			
5 Transformations								•		
6 Eigenvalues				•	•	•			•	•
7 Eigenvectors				•	•	•			•	•
8 Diagonal form									•	•
9 Repeated eigenvalues										•
Interleaved: 2×2 matrices and transformations (FP1)	•	•	•					•		
Interleaved: planes and lines (Chapter 5)							•	•		
Interleaved: vectors, scalar product and cubic equations (P1, P4)				•	•	•			•	•

Exam difficulty. Questions 1 to 5 check each skill on its own. Questions 6 to 10 are multi-step questions built like the last questions on a real WFM03 paper, where one part leads into the next and skills from different topics are combined.

Must learn. Use any zeros when expanding determinants. For eigenvalues, form $\det(\mathbf{M} - \lambda\mathbf{I})$; for eigenvectors, solve $(\mathbf{M} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$; respect any condition such as $k < 0$.

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1.

$$\mathbf{M} = \begin{pmatrix} 2 & 1 & 0 \\ k & 3 & 1 \\ 1 & 0 & 2 \end{pmatrix}$$

(a) Find $\det \mathbf{M}$ in terms of k, and the value of k for which $\mathbf{M}$ is singular.

(4)

(b) Given that $k = 5$, find $\det\left(\mathbf{M}^2\right)$.

(2)

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Question 1 continued

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2.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 2 & 0 & 1 \end{pmatrix}$$

Find $\mathbf{A}^{-1}$, showing the matrix of cofactors.

(6)

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Question 2 continued

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(Total for Question 2 is 6 marks)

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3. **(a)** Given that $\mathbf{A}$ and $\mathbf{B}$ are non-singular 3×3 matrices, prove that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

(2)

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

(b) Find $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$, and hence find $(\mathbf{AB})^{-1}$.

(4)

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Question 3 continued

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4.

$$\mathbf{M} = \begin{pmatrix} 4 & 2 & 0 \\ 1 & 3 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

(a) Show that the eigenvalues of $\mathbf{M}$ are 1, 2 and 5

(3)

(b) Find an eigenvector corresponding to the eigenvalue 5

(3)

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Question 4 continued

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5.

$$\mathbf{M} = \begin{pmatrix} 1 & k & 2 \\ 0 & 3 & 1 \\ 2 & 0 & 2 \end{pmatrix}$$

Given that $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{M}$,

(a) find the corresponding eigenvalue and the value of k,

(4)

(b) find $\det \mathbf{M}$ for this value of k.

(2)

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Question 5 continued

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6.

$$\mathbf{M} = \begin{pmatrix} 2 & 0 & k \\ 0 & 3 & 0 \\ k & 0 & 2 \end{pmatrix}, \qquad k < 0$$

Given that 4 is an eigenvalue of $\mathbf{M}$,

(a) find the value of k,

(3)

(b) find the other two eigenvalues of $\mathbf{M}$,

(3)

(c) find a normalised eigenvector corresponding to the eigenvalue 4,

(2)

(d) explain, without calculating the determinant, why $\mathbf{M}$ is singular.

(2)

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Question 6 continued

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Question 6 continued

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7. Three planes have equations

$$x+y+z=6, \qquad 2x-y+z=3, \qquad x+2y-z=2$$

(a) Write the equations in the form $\mathbf{Mx}=\mathbf{c}$, and find $\mathbf{M}^{-1}$.

(5)

(b) Hence find the point where the three planes meet.

(3)

(c) Explain the significance of $\det \mathbf{M} \neq 0$ for the three planes.

(2)

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Question 7 continued

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Question 7 continued

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8. The transformation T of three-dimensional space is represented by $\mathbf{T} = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

(a) Find $\mathbf{T}^{-1}$.

(5)

(b) The line l has equation $\frac{x-1}{2} = \frac{y}{1} = \frac{z+1}{1}$. Find a vector equation of the image of l under T.

(3)

(c) Find the point whose image under T is $(4, 2, 3)$.

(2)

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Question 8 continued

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Question 8 continued

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9.

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

(a) Find the eigenvalues of $\mathbf{A}$.

(3)

(b) Find a normalised eigenvector for each eigenvalue.

(3)

(c) Write down an orthogonal matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P} = \mathbf{D}$

(3)

(d) Explain why $\mathbf{P}^{-1} = \mathbf{P}^{\mathrm{T}}$

(1)

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Question 9 continued

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Question 9 continued

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10.

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

(a) Show that $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$, and state its eigenvalue.

(2)

(b) Show that the other eigenvalue of $\mathbf{A}$ is 1, repeated.

(3)

(c) Find two perpendicular eigenvectors for the eigenvalue 1, and hence write down an orthogonal matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{\mathrm{T}}\mathbf{AP} = \mathbf{D}$

(4)

(d) Use $\mathbf{D}$ to write down $\det \mathbf{A}$.

(1)

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Question 10 continued

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TOTAL FOR CHAPTER 6 IS 80 MARKS

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