eClassroom

Differentiation

Worked Solutions

Suitable for:

Pearson Edexcel International A-Level (iAL) — Pure Mathematics 2

Question 1 — Worked Solution

(a) Stationary points

$$\frac{dy}{dx} = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3)$$

Set dy/dx = 0: x = 1 or x = 3

$$x = 1: \; y = 1 - 6 + 9 + 2 = 6 \quad \Rightarrow \quad (1,\ 6)$$

$$x = 3: \; y = 27 - 54 + 27 + 2 = 2 \quad \Rightarrow \quad (3,\ 2)$$

$$\therefore \text{Stationary points: } (1,\ 6) \text{ and } (3,\ 2)$$

(b) Nature

$$\frac{d^2y}{dx^2} = 6x - 12$$

At x = 1: $d^2y/dx^2 = -6 < 0$ → **local maximum**

At x = 3: $d^2y/dx^2 = 6 > 0$ → **local minimum**

$$\therefore (1,\ 6) \text{ is a local maximum}; \quad (3,\ 2) \text{ is a local minimum}$$

Question 2 — Worked Solution

(a) Stationary points and nature

$$\frac{dy}{dx} = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x-2)(x+1)$$

Set dy/dx = 0: x = 2 or x = –1

$$x = -1:\ y = -2 - 3 + 12 + 5 = 12 \quad \Rightarrow \quad (-1,\ 12)$$

$$x = 2:\ y = 16 - 12 - 24 + 5 = -15 \quad \Rightarrow \quad (2,\ -15)$$

$$\frac{d^2y}{dx^2} = 12x - 6$$

At x = –1: d^2y/dx^2 = –18 < 0 → **local maximum**

At x = 2: d^2y/dx^2 = 18 > 0 → **local minimum**

$$\therefore (-1,\ 12) \text{ local max;} \quad (2,\ -15) \text{ local min}$$

(b) Decreasing

dy/dx < 0 between the stationary points:

$$\therefore -1 < x < 2$$

Question 3 — Worked Solution

(a) f′(x) factorised

$$f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x-3)(x+1)$$

$$\therefore f'(x) = 3(x-3)(x+1)$$

(b) Increasing

f(x) is increasing where f′(x) > 0, i.e. both factors have the same sign:

$$\therefore x < -1 \quad \text{or} \quad x > 3$$

(c) Decreasing

f(x) is decreasing where f′(x) < 0:

$$\therefore -1 < x < 3$$

Question 4 — Worked Solution

(a) Show dy/dx = 3x² – 10x

$$y = x^2(x-5) = x^3 - 5x^2$$

$$\frac{dy}{dx} = 3x^2 - 10x \quad \checkmark$$

(b) Stationary points

$$3x^2 - 10x = x(3x-10) = 0 \;\Rightarrow\; x = 0 \text{ or } x = \frac{10}{3}$$

$$x = 0:\; y = 0 \quad \Rightarrow \quad (0,\ 0)$$

$$x = \frac{10}{3}:\; y = \frac{100}{9}\left(\frac{10}{3} - 5\right) = \frac{100}{9} \times \left(-\frac{5}{3}\right) = -\frac{500}{27}$$

$$\therefore (0,\ 0) \text{ and } \left(\frac{10}{3},\ -\frac{500}{27}\right)$$

(c) Nature

$$\frac{d^2y}{dx^2} = 6x - 10$$

At x = 0: $d^2y/dx^2 = -10 < 0$ → **local maximum**

At x = 10/3: $d^2y/dx^2 = 20 - 10 = 10 > 0$ → **local minimum**

Question 5 — Worked Solution

(a) dy/dx

$$y = x^2 + 16x^{-1}$$

$$\frac{dy}{dx} = 2x - \frac{16}{x^2}$$

$$\therefore \frac{dy}{dx} = 2x - \frac{16}{x^2}$$

(b) Stationary point and nature

Set dy/dx = 0:

$$2x - \frac{16}{x^2} = 0 \;\Rightarrow\; 2x^3 = 16 \;\Rightarrow\; x^3 = 8 \;\Rightarrow\; x = 2$$

$$y = 4 + \frac{16}{2} = 4 + 8 = 12 \quad \Rightarrow \quad (2,\ 12)$$

$$\frac{d^2y}{dx^2} = 2 + \frac{32}{x^3}$$

At x = 2: $d^2y/dx^2 = 2 + 4 = 6 > 0$ → **local minimum**

$$\therefore (2,\ 12) \text{ is a local minimum}$$

Question 6 — Worked Solution

Find range of k for f(x) always increasing

$$f'(x) = 3x^2 + 2kx + 3$$

f(x) is increasing for all x when f′(x) > 0 for all x. This requires the discriminant of f′(x) to be negative (no real roots, always positive since leading coefficient > 0):

$$\Delta = (2k)^2 - 4(3)(3) < 0$$

$$4k^2 - 36 < 0 \Rightarrow k^2 < 9$$

$$\therefore -3 < k < 3$$

Question 7 — Worked Solution

(a) Show A = 120x – 2x²

Two widths of x m and one length L m use all 120 m of fencing:

$$2x + L = 120 \;\Rightarrow\; L = 120 - 2x$$

$$A = x \times L = x(120 - 2x) = 120x - 2x^2 \quad ✓$$

(b) Maximum area

$$\frac{dA}{dx} = 120 - 4x = 0 \;\Rightarrow\; x = 30$$

$$\frac{d^2A}{dx^2} = -4 < 0 \;\Rightarrow\; \text{maximum confirmed}$$

$$A = 120(30) - 2(900) = 3600 - 1800$$

$$\therefore A_{\max} = 1800\ \text{m}^2$$

(c) Dimensions

Width = 30 m, Length = 120 – 2(30) = 60 m.

$$\therefore 30\ \text{m} \times 60\ \text{m}$$

Question 8 — Worked Solution

(a) Show S = $x^2 + 128/x$

Volume: $x^2h = 32 \Rightarrow h = 32/x^2$

Surface area (no top): one base + four sides:

$$S = x^2 + 4xh = x^2 + 4x \cdot \frac{32}{x^2} = x^2 + \frac{128}{x} \quad ✓$$

(b) Minimum S

$$\frac{dS}{dx} = 2x - \frac{128}{x^2} = 0 \Rightarrow 2x^3 = 128 \Rightarrow x^3 = 64 \Rightarrow x = 4$$

$$\frac{d^2S}{dx^2} = 2 + \frac{256}{x^3}$$

At $x = 4$: $d^2S/dx^2 = 2 + 4 = 6 > 0$ → minimum confirmed

$$\therefore x = 4 \text{ cm}$$

(c) Minimum surface area and height

$$S = 16 + \frac{128}{4} = 16 + 32 = 48 \text{ cm}^2$$

$$h = \frac{32}{16} = 2 \text{ cm}$$

$$\therefore S_{\min} = 48 \text{ cm}^2, \quad h = 2 \text{ cm}$$

Question 9 — Worked Solution

(a) Show V = x(300 – x²)/4

Surface area (no top): one base + four sides:

$$x^2 + 4xh = 300 \;\Rightarrow\; h = \frac{300 - x^2}{4x}$$

$$V = x^2 h = x^2 \times \frac{300 - x^2}{4x} = \frac{x(300 - x^2)}{4} \quad ✓$$

(b) Maximum V

$$V = \frac{300x - x^3}{4}$$

$$\frac{dV}{dx} = \frac{300 - 3x^2}{4} = 0 \;\Rightarrow\; x^2 = 100 \;\Rightarrow\; x = 10$$

$$\frac{d^2V}{dx^2} = \frac{-6x}{4} = -\frac{3x}{2}$$

At x = 10: $d^2V/dx^2 = -15 < 0$ → maximum confirmed

$$\therefore x = 10 \text{ cm}$$

(c) Maximum volume and height

$$V = \frac{10(300 - 100)}{4} = \frac{10 \times 200}{4} = 500 \text{ cm}^3$$

$$h = \frac{300 - 100}{4 \times 10} = \frac{200}{40} = 5 \text{ cm}$$

$$\therefore V_{\text{max}} = 500 \text{ cm}^3, \quad h = 5 \text{ cm}$$

Question 10 — Worked Solution

(a) Derivatives

$$\frac{dy}{dx} = 4x^3 - 24x^2 + 36x = 4x(x-3)^2$$

$$\frac{d^2y}{dx^2} = 12x^2 - 48x + 36 = 12(x-1)(x-3)$$

(b) Stationary points

Set $dy/dx = 0$: $4x(x-3)^2 = 0 \rightarrow x = 0$ or $x = 3$

$$x = 0: \ y = 0 \quad \Rightarrow \quad (0,\ 0)$$

$$x = 3: \ y = 81 - 216 + 162 = 27 \quad \Rightarrow \quad (3,\ 27)$$

$$\therefore \text{Stationary points: } (0,\ 0) \text{ and } (3,\ 27)$$

(c) Nature

At $x = 0$: $d^2y/dx^2 = 36 > 0 \rightarrow$ **local minimum**

At $x = 3$: $d^2y/dx^2 = 0 \rightarrow$ second derivative test is **inconclusive**.

Examine the sign of dy/dx either side of $x = 3$:

$$x = 2.9: \ \frac{dy}{dx} = 4(2.9)(2.9-3)^2 = 4(2.9)(0.01) > 0$$

$$x = 3.1: \ \frac{dy}{dx} = 4(3.1)(3.1-3)^2 = 4(3.1)(0.01) > 0$$

dy/dx does not change sign, so **(3, 27) is a point of inflection**.

$$\therefore (0,\ 0) \text{ is a local minimum;} \quad (3,\ 27) \text{ is a point of inflection}$$