eClassroom

A-LEVEL MATHEMATICS • DECISION 1

Graphs and Networks

Worksheet

Pearson Edexcel International A Level (WDM11) • Calculator allowed: show every step

Free Graphs and Networks worksheet for A-Level and International A-Level Maths, with full worked solutions

A **Examples** & Your turn page 2

B **Fluency** core skills page 4

C **Reasoning** explain it page 7

D **Problem** solving page 8

E **Exam** practice page 12

Tap a section to jump straight to it

By the end of this worksheet you can: use graph vocabulary precisely • degrees and the handshaking lemma • *Stretch:* complete, bipartite and isomorphic graphs, trees, and adjacency and distance matrices.

Which questions are for me?

CORE The methods every A-level student must secure. Start here.

STRETCH The language of graphs used precisely: degrees, paths and cycles, special graphs, isomorphism and matrices.

Every section has both, clearly labelled. Questions get harder as you go, and every skill in Sections A to D is needed in Section E.

Track your progress

Section	B	C	D	E
Core marks	23	10	13	22
Stretch marks	16	14	17	14
My score				

Full worked solutions | Exam questions by topic | Decision 1 exam questions | Formula sheet builder

Name ..... Class ..... Date .....

A Worked examples & Your turn

Read each example carefully. Cover it up, then answer the question beside it using the same method.

CORE *Secure the method*

Worked example A1: degrees

A graph has arcs AB, AC, BC, CD, DE, CE. Find the degree of each vertex.

Count the arcs meeting each vertex

$A2,\ B2,\ C4,\ D2,\ E2$

Your turn A1

A graph has arcs PQ, PR, PS, QR, RS, ST, QT. Find the degree of each vertex.

Worked example A2: the handshaking lemma

The degrees of a graph are 3, 3, 2, 2, 4. How many arcs does it have?

Sum of degrees = 2× arcs: $\frac{14}{2}$

7

Your turn A2

The degrees of a graph are 1, 2, 2, 3, 3, 5. How many arcs does it have?

Worked example A3: path or trail

In the graph with arcs AB, AC, BC, CD, DE, CE, is A–C–D–E–C–B a path?

Every arc exists, but C appears twice

a trail, not a path

Your turn A3

In the graph with arcs PQ, PR, PS, QR, RS, ST, QT, is P–Q–R–S–T a path?

★★★★★

Join students already acing their exams.

Book your FREE lesson at www.eclassroom.org.uk

STRETCH

Worked example A4: complete graphs and trees

Find the number of arcs in K_6, and in a tree with 10 vertices.

K_n: $\frac{1}{2}n(n-1)$; a tree: $n-1$

15 and 9

Your turn A4

Find the number of arcs in K_9, and in a tree with 15 vertices.

Worked example A5: complete bipartite graphs

Find the number of arcs in $K_{2,4}$

Each of 2 vertices joins each of 4

8

Your turn A5

Find the number of arcs in $K_{3,6}$

Worked example A6: an adjacency matrix

How many arcs has the graph with adjacency matrix

	A	B	C	D
A	0	1	1	0
B	1	0	1	1
C	1	1	0	1
D	0	1	1	0

Sum of entries $10 = 2\times$ *arcs*

5

Your turn A6

How many arcs has the graph with adjacency matrix

	A	B	C	D	E
A	0	1	0	1	1
B	1	0	1	0	0
C	0	1	0	1	1
D	1	0	1	0	1
E	1	0	1	1	0

Worked example A7: a distance matrix

A network has AB 5, AC 8, BC 3, BD 6, CD 4. Find its total weight.

26

Your turn A7

A network has AB 7, AD 4, BC 9, BD 2, CD 6, CE 5, DE 8. Find its total weight.

★★★★★
Join students already acing their exams.
Book your FREE lesson at www.eclassroom.org.uk

B Fluency

Each question is a designed sequence. From one part to the next, ask yourself: what changed, and what did it do to the answer?

CORE *Secure the method*

B1 Figure 1 shows a graph.

(a) State the degree of each vertex.

(b) Use the degrees to find the number of arcs.

(c) State how many vertices have odd degree.

(d) State whether the graph is simple and connected.

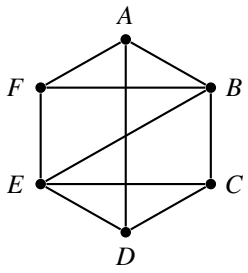


Figure 1

[4 marks]

B2 For the graph in Figure 1,

(a) write down a path from A to C using four arcs

(b) state whether A–B–E–C–D–A is a cycle

(c) state whether A–B–F–E–B–C is a path or a trail

(d) write down a Hamiltonian cycle.

[4 marks]

B3 Find the number of arcs in

(a) K_{10}

(b) $K_{4,6}$

(c) a tree with 12 vertices

(d) K_n, and find n, when it has 66 arcs.

[4 marks]

B4 Simple graphs are to be drawn with the given degrees.

(a) Find the arcs for 2, 2, 3, 3, 4, 4

(b) Explain why 1, 2, 3, 3, 4 is impossible.

(c) Explain why 2, 3, 3, 4, 5 is impossible.

(d) Find the degree sum of a tree with 6 vertices.

[4 marks]

★★★★★
Join students already acing their exams.
Book your FREE lesson at www.eclassroom.org.uk

B5 A graph has adjacency matrix

	A	B	C	D
A	0	1	1	1
B	1	2	1	0
C	1	1	0	1
D	1	0	1	0

(a) State the degree of each vertex.

(b) Find the number of arcs.

(c) State whether the graph is simple and whether it is connected. **[3 marks]**

B6 A network has arcs AB 6, AC 9, AE 4, BC 3, BD 7, CD 5, DE 10.

(a) State the number of arcs.

(b) Find the total weight.

(c) State the degree of each vertex.

(d) Find the shortest route from A to D using two arcs.

[4 marks]

STRETCH

B7 G_1 has arcs AB, BC, CD, DA, AC and G_2 has arcs PQ, QR, RS, SP, QS.

(a) Write down the degrees of each graph.

(b) Give a correspondence showing they are isomorphic.

(c) Explain why every simple graph with 4 vertices and 5 arcs is isomorphic to G_1

(d) State how many arcs K_4 has.

[4 marks]

B8 Ali can do tasks 1 and 2, Ben tasks 2, 3 and 4, and Cara tasks 1 and 4. This is modelled by a bipartite graph.

(a) State the number of arcs.

(b) Show the degree sums of the two sets are equal.

(c) Find the arcs needed to make $K_{3,4}$

(d) Explain why an arc from Ali to Ben would stop the graph being bipartite.

[4 marks]

B9 A connected graph has 9 vertices and 13 arcs.

(a) State the arcs in a spanning tree.

(b) State how many arcs are removed to form it.

(c) Explain why a tree has no cycles.

(d) Find the degree sum of the spanning tree.

[4 marks]

B10 Simple graphs with 8 vertices are considered.

(a) Find the fewest arcs a connected one can have.

(b) Find the most arcs one can have.

(c) Describe the degrees when one arc is missing from K_8

(d) Explain why K_8 has a Hamiltonian cycle.

[4 marks]

C Reasoning

Show your working and explain your thinking. A correct answer without a reason does not score full marks.

CORE *Secure the method*

C1 In a graph with arcs AB, BC, CA and AD, Amir calls A–B–C–A–D a path. Explain his error. **[2 marks]**

C2 Bea says she can draw a graph with three vertices, each of degree 3. Explain why she cannot. **[2 marks]**

C3 A graph has degrees 4, 3, 3, 2, 2 and Cal says it has 14 arcs. Explain his error and find the number of arcs. **[3 marks]**

C4 A vertex is joined to two other vertices and has a loop. Dev gives its degree as 3. Explain his error, and state how the loop appears in the adjacency matrix. **[3 marks]**

STRETCH

C5 Eve says that two graphs with the same degrees must be isomorphic. Use two graphs with six vertices, each of degree 2, to show that she is wrong. **[3 marks]**

C6 Fay says a tree with 8 vertices has 8 arcs. Explain her error, and what an eighth arc would create. **[3 marks]**

C7 Gus says $K_{3,3}$ has 6 arcs because "there are 3 and 3 vertices". Explain his error and give the correct number. **[3 marks]**

C8 **(a)** Prove the handshaking lemma: the sum of the degrees of a graph is twice the number of arcs. **(2)**

(b) Hence prove that every graph has an even number of vertices of odd degree. **(3)**

[5 marks]

★★★★★ **Join students already acing their exams.**
Book your FREE lesson at www.eclassroom.org.uk

D Problem solving

You decide the method. Read the whole question first, and show every step.

CORE *Secure the method*

D1 Figure 2 shows a graph G with six vertices representing villages, and arcs representing the roads between them.

(a) State the degree of each vertex and hence the number of arcs in G

(1)

(b) A planner claims a different road map of six villages could have vertex degrees 1, 2, 2, 3, 3 and 4. Explain why this is impossible.

(1)

(c) Determine the number of extra roads needed for every pair of villages in G to be joined directly.

(1)

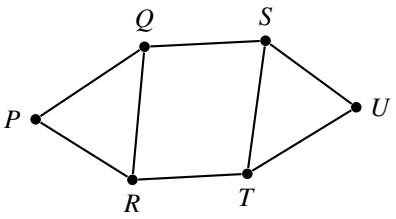


Figure 2

[3 marks]

D2 Figure 3 shows a graph T with six vertices.

(a) Write down a path from A to D that uses exactly three arcs.

(1)

(b) State, with a reason, whether A–B–F–D–E–A is a cycle on T

(1)

(c) Write down a Hamiltonian cycle on T

(1)

(d) State the number of arcs in T, and explain how the degrees confirm it.

(1)

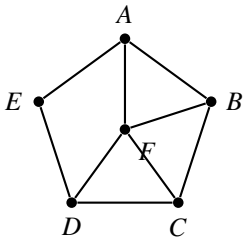


Figure 3

[4 marks]

D3 Figure 4 shows a graph with four vertices, a double arc between A and B and a loop at D.

(a) Write down the adjacency matrix for the graph.

(2)

(b) State the sum of the entries in your matrix, and explain why the graph is not simple.

(1)

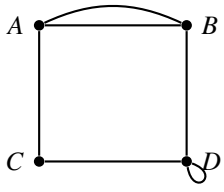


Figure 4

[3 marks]

D4 Figure 5 shows a network of footpaths between eight points in a park. The number on each arc is the length of the footpath in metres.

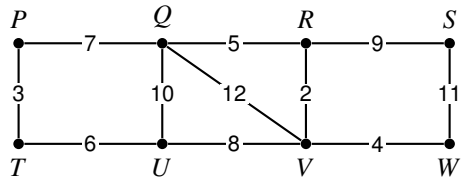


Figure 5

(a) Find the total length of the footpaths.

(1)

(b) A spanning tree is to be chosen from the footpaths. State how many arcs must be removed.

(1)

(c) State, with a reason, whether P–Q–V–U–T–P is a cycle, and give its length.

(1)

[3 marks]

STRETCH

D5 Figures 6, 7 and 8 show three graphs, each with six vertices and eight arcs.

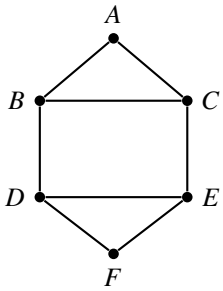


Figure 6

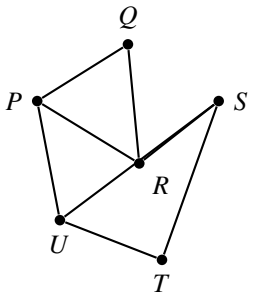


Figure 7

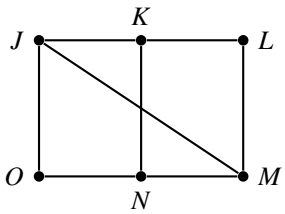


Figure 8

(a) Show that the graphs in Figure 6 and Figure 7 are isomorphic by giving a complete correspondence between their vertices.

(2)

(b) The graph in Figure 8 has the same vertex degrees as the graph in Figure 6. Explain why it is not isomorphic to it.

(2)

[4 marks]

D6 Figure 9 shows which of three students, W, X and Y, belong to which of five clubs, 1 to 5.

(a) Explain why the graph is bipartite, and state the number of arcs.

(1)

(b) Show that the sum of the degrees of the students equals the sum of the degrees of the clubs.

(1)

(c) Find how many more arcs would make the graph $K_{3,5}$

(1)

(d) Explain why adding an arc from W to X would stop the graph being bipartite.

(1)

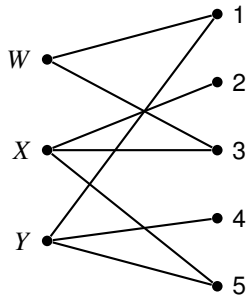


Figure 9

[4 marks]

D7 The table shows the lengths, in km, of the direct roads between six depots. A dash shows there is no direct road.

	A	B	C	D	E	F
A	–	4	7	–	–	–
B	4	–	–	5	10	–
C	7	–	–	6	9	–
D	–	5	6	–	–	8
E	–	10	9	–	–	3
F	–	–	–	8	3	–

(a) State the number of arcs in the network, and find its total weight.

(1)

(b) State the degree of each depot, and the number of depots of odd degree.

(1)

(c) Determine the fewest roads that any route from A to F must use, giving a reason.

(2)

[4 marks]

D8 Figure 10 shows a network of seven towns. The number on each arc is the travel time in minutes.

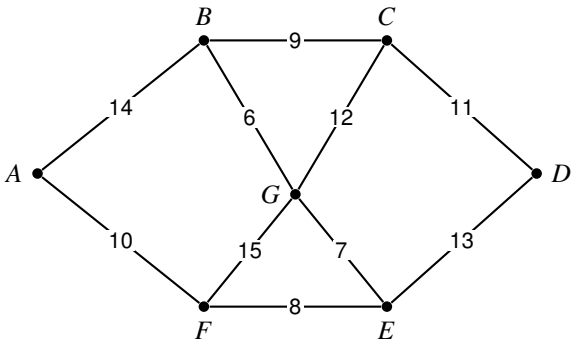


Figure 10

(a) State the degree of each vertex and find the total weight of the network.

(2)

(b) Find every Hamiltonian cycle that starts and finishes at A, giving the length of each, and hence state the shortest.

(3)

[5 marks]

E Exam practice

Exam-style questions in the style of WDM11. Write your answers in the spaces, and show every step.

CORE *Exam questions for everyone*

E1 Figure 11 shows a network of roads between eight towns, A, B, C, D, E, F, G and H. The number on each arc is the length, in km, of the corresponding road. A delivery company is studying the network before planning its routes. It wants to understand how the towns are connected, which sequences of towns can be followed along the roads, and whether a single circular tour can visit every town exactly once before returning to its starting town.

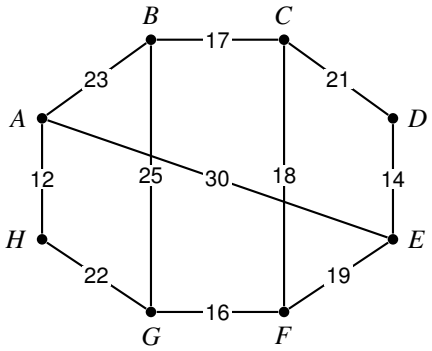


Figure 11

(a) State the number of arcs in the network, and use the degrees of the vertices to confirm your answer. Hence find the total length of the roads.

(2)

(b) For each of the following, state whether it is a path, a trail or a cycle on the network, giving a reason.

(i) A–B–C–D–E–F–G–H (ii) A–B–G–F–C–B (iii) A–E–F–C–B–G–H–A

(2)

(c) By considering the vertices of degree 2, explain why the network has only one Hamiltonian cycle, and find its length. You must make clear which arcs are forced into the cycle and why the remaining arcs cannot be used.

(4)

(d) A spanning tree for the network is to be found. State the number of arcs in a spanning tree, and the number of roads that would not be used.

(2)

(Total for Question E1 is 10 marks)

Answer book: Question E1

Work in the space on this page, as you would in the WDM11 answer book.

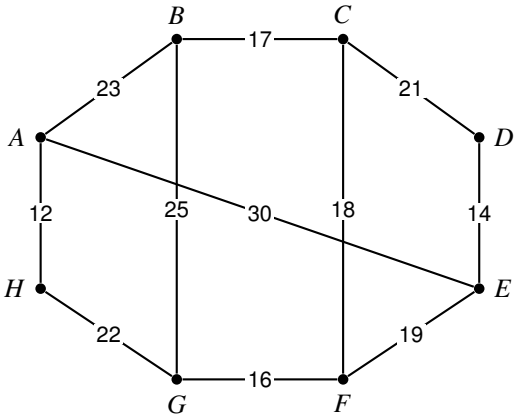


Figure 11 (printed again for you to mark up)

★★★★★
Join students already acing their exams.
Book your FREE lesson at www.eclassroom.org.uk

E2 The table shows the distances, in miles, along the direct roads between seven towns, J, K, L, M, N, P and R. A dash shows that there is no direct road. Figure 12 shows the positions of the towns on a map. A regional council is using a graph to model the roads, so that it can describe the network accurately in a report and compare it with other possible road layouts.

	J	K	L	M	N	P	R
J	–	8	13	–	–	–	–
K	8	–	6	11	–	–	–
L	13	6	–	–	9	–	–
M	–	11	–	–	4	10	–
N	–	–	9	4	–	7	12
P	–	–	–	10	7	–	5
R	–	–	–	–	12	5	–

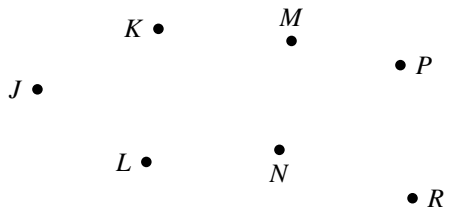


Figure 12

(a) Using the positions in the answer book, draw the network of roads described by the table, showing the distance on each arc.

(3)

(b) State the number of arcs and find the total length of the roads.

(1)

(c) State the degree of each town, and explain how your answer to (b) confirms the handshaking lemma.

(2)

(d) Find a Hamiltonian cycle on the network, and its length. Explain why both arcs at J and both arcs at R must be used in any Hamiltonian cycle on this network.

(3)

(e) Determine how many more roads would be needed for every pair of towns to be joined directly.

(1)

(f) Explain why the network is not a tree, and state how many roads would need to be removed to leave a spanning tree.

(2)

(Total for Question E2 is 12 marks)

Answer book: Question E2

Work in the space on this page, as you would in the WDM11 answer book.

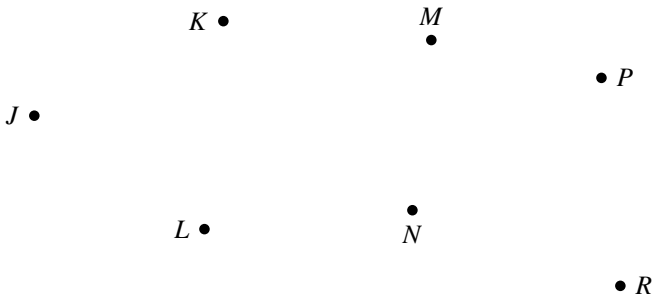


Diagram 1 (draw the network here)

★★★★★

Join students already acing their exams.
Book your FREE lesson at www.eclassroom.org.uk

STRETCH

E3 Figure 13 shows a graph G drawn as two squares, one inside the other, with each corner of the outer square joined to the corresponding corner of the inner square. Figure 14 shows a graph H drawn as an octagon with four chords. Two students disagree about whether these drawings represent the same graph, since they look very different on the page. In this question you will settle the disagreement and investigate further properties of G

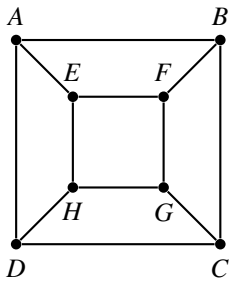


Figure 13

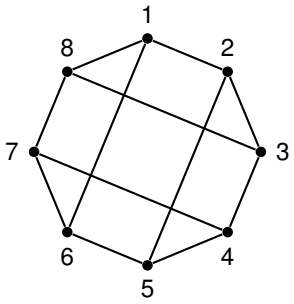


Figure 14

(a) State the degree of every vertex of G, and use the handshaking lemma to find the number of arcs.

(2)

(b) Show that G and H are isomorphic. You must give a complete correspondence between their vertices and explain how you checked it.

(4)

(c) Show that G is bipartite by dividing its vertices into two sets of four, explaining how you know that no arc joins two vertices in the same set.

(3)

(d) Determine how many arcs would need to be added to G to make the complete bipartite graph on your two sets.

(2)

(e) Write down a Hamiltonian cycle on G

(1)

(f) Explain, using the number of arcs, why G is not a tree.

(2)

(Total for Question E3 is 14 marks)

★★★★★
Join students already acing their exams.
Book your FREE lesson at www.eclassroom.org.uk

Answer book: Question E3

Work in the space on this page, as you would in the WDM11 answer book.

Complete this table

G	A	B	C	D	E	F	G	H
H								

Want more exam practice? The **Exam Questions by Topic** booklets have 10 more full exam questions on Decision 1 Chapter 2, Graphs and Networks, with a full mark scheme. **Open the booklet**

Finished? Mark your work with the **full worked solutions**, fill in your scores on the front page, then practise real exam questions on the **Decision 1 exam questions page**.