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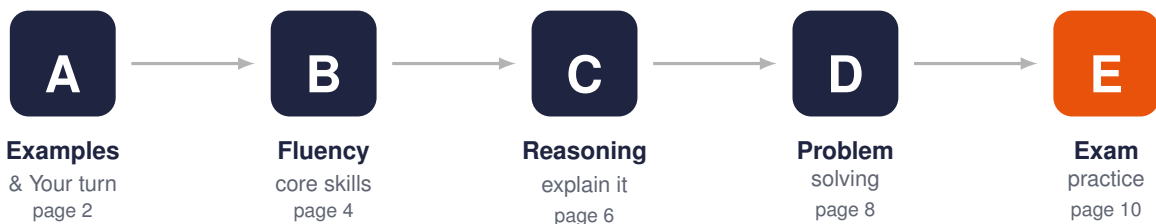
A-LEVEL MATHEMATICS • PURE 2

Proof

Worksheet

Pearson Edexcel International A Level (WMA12) & A Level (9MA0) • Calculator allowed: show every step

Free Proof worksheet for A-Level and International A-Level Maths, with full worked solutions



Tap a section to jump straight to it

By the end of this worksheet you can: write numbers algebraically and prove results by deduction • prove results by exhaustion, checking every case and writing a conclusion • disprove a statement with a single counter-example • use completing the square to prove an expression is always positive • *Stretch:* justify each step of an inequality proof, choose cases by remainders and decide whether a result needs its conditions.

Which questions are for me?

CORE The methods every A-level student must secure.
Start here.

STRETCH Proofs that need a sign stated before dividing, counter-examples that keep the condition, and exhaustion over every case.

Every section has both, clearly labelled. Questions get harder as you go, and every skill in Sections A to D is needed in Section E.

Track your progress

Section	B	C	D	E
Core marks	23	10	13	11
Stretch marks	16	14	17	8
My score				

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A**Worked examples & Your turn**

Read each example carefully. Cover it up, then answer the question beside it using the same method.

CORE*Secure the method***Worked example A1**

Prove that the sum of any two consecutive odd numbers is a multiple of 4

Write the numbers in general: $2n + 1$ and $2n + 3$, where n is an integer.

$$(2n + 1) + (2n + 3) = 4n + 4 = 4(n + 1)$$

$n + 1$ is an integer, so finish with a statement.

a multiple of 4, as required

Your turn A1

Prove that the product of any two consecutive even numbers is a multiple of 8

Worked example A2

Prove by exhaustion that $n^2 + n$ is even for every integer n

Every integer is even or odd: two cases.

$$n = 2k: 4k^2 + 2k = 2(2k^2 + k)$$

$$n = 2k + 1: (2k + 1)(2k + 2) = 2(2k + 1)(k + 1)$$

even in both cases, so always even

Your turn A2

Prove by exhaustion that, for every integer n , n^2 is either a multiple of 3 or one more than a multiple of 3

Worked example A3

Disprove the statement: "if $x^2 > 9$ then $x > 3$ "

Find one value that makes the condition true but the conclusion false.

$$x = -4: x^2 = 16 > 9, \text{ but } -4 < 3$$

$x = -4$ is a counter-example

Your turn A3

Disprove the statement: " $2^n + 1$ is prime for every positive integer n "



STRETCH

Worked example A4: completing the square

Prove that $x^2 - 6x + 11 > 0$ for all real x

$$x^2 - 6x + 11 = (x - 3)^2 + 2$$

A square is never negative: $(x - 3)^2 \geq 0$

$$(x - 3)^2 + 2 \geq 2 > 0$$

Your turn A4

Prove that $n^2 - 8n + 20 > 0$ for all real n

Worked example A5: state the sign before dividing

a and b are positive, and $a^2 > b^2$. Prove that $a > b$

$$a^2 - b^2 = (a - b)(a + b) > 0$$

$a + b > 0$ because a and b are positive, so dividing by it keeps the inequality.

$$a - b > 0, \text{ so } a > b$$

Your turn A5

p and q are positive, and $\frac{p}{q} > \frac{q}{p}$. Prove that $p > q$

Worked example A6: testing for a prime

Prove that 13 is a prime number.

A factor pair of 13 must include a number up to $\sqrt{13} \approx 3.6$, so test the primes 2 and 3

$$13 \text{ is odd, and } 13 = 3 \times 4 + 1$$

no factor, so 13 is prime

Your turn A6

Prove that 37 is a prime number.

Worked example A7: counter-examples must keep the condition

Disprove: "for all real x and y , if $x > y$ then $x^2 > y^2$ "

Choose $x > y$, but with y negative.

$$x = 1, y = -2: 1 > -2, \text{ but } 1 < 4$$

$$x = 1, y = -2$$

Your turn A7

Disprove: "for all non-zero real a and b , if $a > b$

then $\frac{1}{a} < \frac{1}{b}$ "



B**Fluency**

Each question is a designed sequence. From one part to the next, ask yourself: what changed, and what did it do to the answer?

CORE*Secure the method*

B1 Write down an expression, in terms of an integer n , for

- | | |
|--------------------------------|--|
| (a) any even number | (b) any odd number |
| (c) three consecutive integers | (d) any number that is one more than a multiple of 3 |

[4 marks]

B2 Prove that

- (a) the sum of any two odd numbers is even
- (b) the product of any two odd numbers is odd
- (c) the sum of any three consecutive integers is a multiple of 3
- (d) one less than the square of any odd number is a multiple of 4

[4 marks]

B3 Disprove each statement with a counter-example.

- | | |
|--|---|
| (a) All prime numbers are odd. | (b) $x^2 \geq x$ for all real x |
| (c) $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$ for all positive a, b | (d) $n^2 - n + 11$ is prime for all positive integers n |

[4 marks]

B4 By completing the square, prove that, for all real x ,

- | | |
|-------------------------|--------------------------|
| (a) $x^2 + 4x + 5 > 0$ | (b) $x^2 - 10x + 26 > 0$ |
| (c) $2x^2 - 4x + 3 > 0$ | (d) $-x^2 + 6x - 10 < 0$ |

[4 marks]



CORE*continued: exhaustion and true or false***B5** Prove each statement by exhaustion.**(a)** $n^2 + 3$ is not a multiple of 5 for $n = 1, 2, 3, 4, 5$ **(b)** $p^2 - 1$ is a multiple of 24 for every prime p with $3 < p < 20$ **(c)** $3^n + 1$ is even for $n = 1, 2, 3, 4$ **[3 marks]****B6** State whether each statement is true or false. Give a proof if it is true, and a counter-example if it is false.**(a)** The sum of any two prime numbers is even.**(b)** The product of any two consecutive integers is even.**(c)** The square of any odd number is odd.**(d)** $x^3 > x$ for all $x > 0$ **[4 marks]****STRETCH****B7** Use algebra to prove each result by exhaustion, for every integer n **(a)** n^2 is a multiple of 4 or one more than a multiple of 4**(b)** $n^2 + 1$ is never a multiple of 3**(c)** $n^3 + 2n$ is a multiple of 3**(d)** $n(n+1)(2n+1)$ is a multiple of 6**[4 marks]****B8** Given that a and b are positive and $x > 0$, prove that**(a)** $a^2 + b^2 \geq 2ab$ **(b)** $\frac{a}{b} + \frac{b}{a} \geq 2$ **(c)** $\frac{a+b}{2} \geq \sqrt{ab}$ **(d)** $x + \frac{1}{x} \geq 2$ **[4 marks]****B9** Disprove each statement with a counter-example that satisfies its condition.**(a)** If $x^2 = 4$ then $x = 2$ **(b)** If n is odd then $n^2 + 1$ is a multiple of 4**(c)** If p and q are irrational then $p + q$ is irrational.**(d)** If x is a positive integer then $2^x > x^2$ **[4 marks]****B10** In each part, use the smallest possible set of prime factors.**(a)** Prove that 41 is prime.**(b)** Prove that 97 is prime.**(c)** Show that 91 is not prime.**(d)** Explain why, to test whether N is prime, only primes up to $\sqrt{N}$ need to be tried.**[4 marks]**

C**Reasoning**

Show your working and explain your thinking. A correct answer without a reason does not score full marks.

CORE*Secure the method*

- C1** Sam is asked to prove that the sum of any two odd numbers is even. He writes: “ $3 + 5 = 8$, $7 + 9 = 16$ and $11 + 13 = 24$, which are all even.”
Explain why this is not a proof. [2 marks]
- C2** Lily proves that the product of any two odd numbers is odd by writing them as $2n + 1$ and $2n + 3$
Explain why her proof is not complete, and say how to fix it. [2 marks]
- C3** Kai proves by exhaustion that $n^2 + n + 1$ is odd for every integer n . He writes:
$$n = 2k: 4k^2 + 2k + 1 = 2(2k^2 + k) + 1, \text{ which is odd.}$$

Explain what is missing from Kai's proof, and complete it. [3 marks]
- C4** To disprove “for all non-zero real x and y , if $x > y$ then $\frac{1}{x} < \frac{1}{y}$ ”, Jo offers $x = 1$, $y = 2$, because $\frac{1}{1} > \frac{1}{2}$
Explain why Jo's counter-example is not valid, and give a valid one. [3 marks]



STRETCH

- C5** a and b are positive, and $a^3b > ab^3$. Max writes: "Divide by ab : $a^2 > b^2$, so $a > b$."
Max's conclusion is correct. Explain which reasons are missing from his proof, and write the full proof. [3 marks]
- C6** A student says: "If p is prime, then $2^p - 1$ is prime."
Show that the statement is true for $p = 2, 3, 5$ and 7 , and then show that it is false, using $2^{11} - 1 = 2047$ [3 marks]
- C7** Prove that $\sqrt{3} + \sqrt{5} < 4$, explaining why squaring both sides is valid here. [3 marks]
- C8** (a) Prove that $n^3 - n$ is a multiple of 6 for every integer n (2)
- (b) Show, by a counter-example, that $n^3 - n$ is not always a multiple of 12 (1)
- (c) Prove that $n^3 - n$ is a multiple of 24 for every odd integer n (2)
- [5 marks]



D**Problem solving**

You decide the method. Read the whole question first, and show every step.

CORE*Secure the method*

- D1** Use algebra to prove by exhaustion that the square of any integer is either a multiple of 3, or one more than a multiple of 3, by considering the cases $n = 3k$, $n = 3k + 1$ and $n = 3k + 2$, where k is an integer. Show your working clearly and finish with a conclusion. **[3 marks]**
- D2** A student claims that " $n^2 + n + 17$ is prime for every positive integer n "
- (a) Show, by a counter-example, that the student's claim is false. **(1)**
- (b) Prove by exhaustion that $n^2 + n + 17$ is odd for every integer n , showing clearly the two cases you use and stating a conclusion. **(3)**
- [4 marks]**
- D3** a and b are positive numbers such that $\frac{a}{b} + \frac{b}{a} = 2$
- Prove that $a = b$, explaining each step, including why it is valid to multiply both sides by ab **[3 marks]**
- D4** By completing the square, prove that $x^2 - 4x + 7 \geq 3$ for all real x , and state the value of x for which equality holds. **[3 marks]**



STRETCH

D5 In this question you must show detailed reasoning.

(a) Given that x and y are positive numbers such that $x^3 - y^3 < x^2y - xy^2$ prove that $x < y$

(3)

(b) Kai says that the result in part (a) is true for all real numbers x and y , not just positive ones. Is Kai correct? Give a reason for your answer, referring to your working in part (a).

(1)

[4 marks]

D6 (a) By writing an odd number as $2k + 1$, where k is an integer, prove that the square of any odd number is one more than a multiple of 8

(3)

(b) Hence prove that the difference between the squares of any two odd numbers is a multiple of 8

(1)

[4 marks]

D7 (a) Explain why, to decide whether 221 is prime, it is enough to test the primes up to 14

(1)

(b) Show that 221 is not prime.

(1)

(c) Prove by exhaustion that 223 is prime, showing which primes you test.

(2)

[4 marks]

D8 In this question n is an integer.

(a) Prove that, for every integer n ,

$$n(n+1)(n+2)(n+3) + 1 = (n^2 + 3n + 1)^2$$

(3)

(b) Hence write $5 \times 6 \times 7 \times 8 + 1$ as the square of an integer.

(1)

(c) Show, by a counter-example, that the product of three consecutive positive integers plus 1 is not always a square number.

(1)

[5 marks]



**Exam practice**

Exam-style questions in the style of WMA12. Write your answers in the spaces, and show every step.

CORE

Exam questions for everyone

- E1** (i) Use a counter-example to show that the following statement is false.

“ $n^2 + n + 1$ is prime for all positive integers n ”

.....
(2)

- (ii) Use algebra to prove by exhaustion that, for all integers n ,
 $n^3 + n$ is even

showing each case clearly.

(3)

(Total for Question E1 is 5 marks)

- E2** In this question you must show detailed reasoning.

- (a) Given that x and y are positive numbers such that $x > y$, prove that

$$\frac{x}{y} > \frac{x+1}{y+1}$$

(4)

- (b) Using a counter-example, show that the result in part (a) is not true for all real numbers x and y with $x > y$ and $y \neq 0, -1$

(2)

(Total for Question E2 is 6 marks)



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STRETCH

E3 (a) By considering the last digit of an integer n , prove by exhaustion that the last digit of n^2 is always 0, 1, 4, 5, 6 or 9. You may use a table.

(4)

(b) Hence prove that there is no integer n for which $n^2 = 10m + 3$, where m is an integer.

(1)

(c) Show, by a counter-example, that a positive integer whose last digit is 6 is not necessarily a square number.

(1)

(d) Prove that the last digit of n^4 is always 0, 1, 5 or 6

(2)

(Total for Question E3 is 8 marks)

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