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A-LEVEL MATHEMATICS • PURE 2

# Integration

Worked Solutions

Pearson Edexcel International A Level (WMA12) & A Level (9MA0) • Calculator allowed: show every step

Full worked solutions for the free Integration worksheet, A-Level and International A-Level Maths



Tap a section to jump straight to it

**Full worked solutions** for every question on the Pure 2 Integration worksheet, in the same order and with the same question numbers.

**How to mark your work.** Tick each step you got right, not just the final answer. At A-level most marks are for method, so a wrong final answer can still earn most of the marks. If you lost marks, find the *first* line where your working is different from ours. That is the thing to fix.

In Section E: **M** = method mark, **dM** = method mark that depends on the previous M mark, **A** = accuracy mark (needs the M mark before it), **B** = independent mark.

Question sheet

Exam questions by topic

Integration revision engine

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**A****Your turn answers**

Compare each line of your working with the solution.

**CORE**

**A1**  $\left[x^4 - 4x^{\frac{3}{2}}\right]_1^4 = (256 - 32) - (1 - 4) = \boxed{227}$

**A2** Limits 0 and 5:  $\left[\frac{5}{2}x^2 - \frac{1}{3}x^3\right]_0^5 = \frac{125}{2} - \frac{125}{3} = \boxed{\frac{125}{6}}$

**A3**  $h = 0.5$ ; ordinates 1, 1.7321, 3, 5.1962, 9:  $0.25(1 + 9 + 2 \times 9.9282) \approx \boxed{7.464}$

**STRETCH**

**A4**  $\left[\frac{1}{4}x^4 - \frac{9}{2}x^2\right]_0^3 = \frac{81}{4} - \frac{81}{2} = -\frac{81}{4}$ ; below the axis, so the area is  $\boxed{\frac{81}{4}}$

**A5**  $x^2 = x + 6$  at  $x = -2$  and  $x = 3$ ; the line is on top:  $\int_{-2}^3 (x + 6 - x^2) dx = \frac{27}{2} - \left(-\frac{22}{3}\right) = \boxed{\frac{125}{6}}$

**A6**  $[x^2 - 3x]_1^k = k^2 - 3k + 2 = 6$ , so  $k^2 - 3k - 4 = 0$ ,  $(k - 4)(k + 1) = 0$ :  $\boxed{k = 4}$

**A7** The curve bends downwards (concave), so each trapezium lies below it: **an underestimate**



## B

## Fluency solutions

One mark for each correct part. The notes in *italics* explain why each sequence was chosen.

## CORE

B1 (a)  $\boxed{8}$  (b)  $\boxed{7}$  (c)  $8 - (-1) = \boxed{9}$  (d)  $\boxed{-8}$

*Swapping the limits changes the sign.*

B2 (a)  $\left[\frac{2}{3}x^{\frac{3}{2}}\right]_1^4 = \boxed{\frac{14}{3}}$  (b)  $\left[2x^{\frac{1}{2}}\right]_1^4 = \boxed{2}$  (c)  $\left[-2x^{-1}\right]_1^2 = \boxed{1}$  (d)  $\left[x - 2x^{-1}\right]_1^4 = \boxed{\frac{9}{2}}$

*Rewrite as powers of  $x$  before integrating.*

B3 (a)  $\boxed{36}$  (b) integral  $-36$ :  $\boxed{36}$  (c)  $\boxed{18}$  (d) limits 1 and 4:  $\boxed{\frac{9}{2}}$

*An area is never negative; a symmetrical region can be found from half of it.*

B4 (a)  $\boxed{0.5}$  (b)  $0.25(1 + 5 + 2(1.25 + 2 + 3.25)) = \boxed{4.75}$  (c)  $\boxed{\frac{14}{3}}$  (d) the curve is convex, so the trapezia

lie above it:  $\boxed{\text{too large}}$

*Comparing with the exact value shows the size of the error.*

B5 (a)  $\boxed{x = \pm 2}$  (b)  $\boxed{\frac{16}{3}}$  (c) rectangle  $4 \times 4 = 16$ :  $16 - \frac{16}{3} = \boxed{\frac{32}{3}}$

*The area between is the rectangle minus the area under the curve.*

B6 (a)  $\boxed{-\frac{4}{3}}$  (b)  $\boxed{\frac{4}{3}}$  (c)  $\boxed{0}$  (d)  $\frac{4}{3} + \frac{4}{3} = \boxed{\frac{8}{3}}$

*Parts below the axis cancel parts above in one integral: split at the root.*

## STRETCH

B7 (a)  $k^2 = 25$ :  $\boxed{5}$  (b)  $k^3 = 64$ :  $\boxed{4}$  (c)  $k^2 + 3k - 4 = 14$ ,  $(k+6)(k-3) = 0$ :  $\boxed{3}$  (d)  $\frac{2}{3}(8 - k^{\frac{3}{2}}) = \frac{14}{3}$ ,  $k^{\frac{3}{2}} = 1$ :  $\boxed{1}$

*Evaluate in terms of  $k$ , then use the condition on  $k$  to choose the root.*

B8 (a)  $\boxed{20}$  (b)  $10 + 2 = \boxed{12}$  (c)  $\boxed{-10}$  (d)  $10 - \left[\frac{3}{2}x^2\right]_1^3 = 10 - 12 = \boxed{-2}$

*A constant integrates to (constant)  $\times$  (width); the limits of  $\int_1^3 1 \, dx$  give 2*



**B9** (a)  $x^2 - 3x = 0$ :  $x = 0, x = 3$  (b) at  $x = 1$ : line 1, curve  $-1$ : **the line**

(c)  $\int_0^3 (3x - x^2) dx$  (d)  $\frac{27}{2} - 9 = \frac{9}{2}$

Test one point between the limits to see which graph is higher.

**B10** (a)  $h = 1$ :  $0.5(0 + 8 + 2) = 5$  (b)  $h = 0.5$ :  $0.25(0 + 8 + 2 \times 4.5) = 4.25$  (c) **4**

(d) Both overestimate, because the curve is convex, and **more strips give a better estimate**

## C

### Reasoning solutions

Your wording can be different. Check that your reason makes the same point.

#### CORE

**C1** The value at the upper limit comes first:  $[x^2]_1^3 = 9 - 1 = 8$

**C2** The region is below the  $x$ -axis, so the integral is negative; an area is always positive:

$$\frac{32}{3}$$

**C3** The curve is below the axis for  $0 < x < 1$ , so that part cancels some of the rest. Split at  $x = 1$ :

$$\int_0^1 (x^2 - 1) dx = -\frac{2}{3} \text{ and } \int_1^2 (x^2 - 1) dx = \frac{4}{3}: \frac{2}{3} + \frac{4}{3} = 2$$

**C4** The strip width is  $\frac{3-1}{4} = 0.5$ , not  $\frac{3}{4}$

Ordinates 1, 2.25, 4, 6.25, 9:  $0.25(1 + 9 + 2 \times 12.5) = 8.75$

#### STRETCH

**C5** Whether the rule over- or underestimates depends on the shape (the bend) of the curve, not on whether it increases.

$y = x^3$  bends upwards for  $x > 0$ , so the trapezia lie above it: **the estimate is too large**

**C6** Ali has found the area down to the  $x$ -axis; the region stops at  $y = 2$ , so the rectangle  $2 \times 2$  must be subtracted.

$$\frac{28}{3} - 4 = \frac{16}{3}$$



**C7** The region below the axis ( $-2 \leq x \leq 0$ ) cancels the region above in a single integral.

Each region has area  $\int_0^2 x^3 dx = 4$ : total **8**

**C8** (a)  $\int_0^a x^2 dx = \frac{a^3}{3}$ , and the rectangle has area  $a \times a^2 = a^3$  **as required**

(b)  $\int_0^a x^n dx = \frac{a^{n+1}}{n+1}$ , and the rectangle has area  $a \times a^n = a^{n+1}$ , so the ratio is  $\frac{1}{n+1}$  **as required**

## D

### Problem solving solutions

There is often more than one correct method. Any complete, correct method scores full marks.

#### CORE

**D1**  $\int_1^4 (6x^{\frac{1}{2}} - 2x^{-2}) dx = \left[ 4x^{\frac{3}{2}} + 2x^{-1} \right]_1^4$   
 $= \left( 32 + \frac{1}{2} \right) - (4 + 2) = \frac{53}{2}$

**D2** (a)  $x^2 - 2x - 3 = 0$ ,  $(x-3)(x+1) = 0$ :  $x = -1$ ,  $x = 3$

(b)  $\left[ 3x + x^2 - \frac{1}{3}x^3 \right]_{-1}^3 = (9 + 9 - 9) - \left( -3 + 1 + \frac{1}{3} \right) = 9 + \frac{5}{3} = \frac{32}{3}$

**D3** (a) **3.7647** and **2.56**

(b)  $\frac{0.25}{2} (4 + 2 + 2(3.7647 + 3.2 + 2.56)) \approx \mathbf{3.1312}$

**D4**  $8 - x^2 = 2x$ , so  $x^2 + 2x - 8 = 0$ ,  $x = -4$  or  $2$

$\int_{-4}^2 (8 - x^2 - 2x) dx = \left[ 8x - \frac{1}{3}x^3 - x^2 \right]_{-4}^2 = \frac{28}{3} - \left( -\frac{80}{3} \right) = \mathbf{36}$

#### STRETCH

**D5**  $y = 4x^{\frac{1}{2}} - x^{\frac{3}{2}}$ :  $\left[ \frac{8}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}} \right]_0^4 = \frac{64}{3} - \frac{64}{5} = \frac{128}{15}$



**D6** (a)  $(x-1)(x-5) = 0$ :  $x=1$ ,  $x=5$

(b) One integral would let  $R_2$  (negative) cancel  $R_1$ .  $\int_0^1 (x^2 - 6x + 5) dx = \frac{7}{3}$  and  $\int_1^5 (x^2 - 6x + 5) dx = -\frac{32}{3}$ , so the total area is  $13$

**D7** (a)  $R_1$  is below the axis, so its integral is  $-A$ ;  $R_2$  is above, so its integral is  $+A$ . Together they give 0

(b)  $\frac{1}{3}k^3 - \frac{3}{2}k^2 = 0$ , so  $k^2 \left( \frac{1}{3}k - \frac{3}{2} \right) = 0$ , and  $k > 3$ :  $k = \frac{9}{2}$

**D8** (a)  $\frac{dy}{dx} = 2x = 4$  at  $x = 2$ :  $y - 4 = 4(x - 2)$ :  $y = 4x - 4$ ;  $y = 0$  gives  $T(1, 0)$

(b) Area under  $C$  from 0 to 2:  $\frac{8}{3}$ . Triangle under  $l$ :  $\frac{1}{2} \times 1 \times 4 = 2$

$$R = \frac{8}{3} - 2 = \frac{2}{3}$$

## E

## Exam practice solutions

Mark-scheme notes are shown on the right of each line.

## CORE

Exam questions for everyone

**E1** (a)  $\frac{x^2 + 4x + 4}{x^{\frac{1}{2}}} = x^{\frac{3}{2}} + 4x^{\frac{1}{2}} + 4x^{-\frac{1}{2}}$ :  $p = \frac{3}{2}, q = \frac{1}{2}, r = -\frac{1}{2}$

M1 A1

(b)  $\left[ \frac{2}{5}x^{\frac{5}{2}} + \frac{8}{3}x^{\frac{3}{2}} + 8x^{\frac{1}{2}} \right]_1^4$

M1 A1

$$= \left( \frac{64}{5} + \frac{64}{3} + 16 \right) - \left( \frac{2}{5} + \frac{8}{3} + 8 \right) = \frac{586}{15}$$

A1

**E2** (a)  $1.4142$

B1

(b)  $\frac{0.5}{2} (1 + 3 + 2(1.0607 + 1.4142 + 2.0917))$

B1 M1

$$3.283$$

A1

(c) (i)  $\int_0^2 5 dx - 3.283 = 10 - 3.283 = 6.717$

B1ft

(ii) a translation of the curve 1 to the right:  $3.283$

B1ft



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## STRETCH

- E3** (a)  $x(x-2)^2 = x$ , so  $x((x-2)^2 - 1) = 0$  M1  
 $x(x-1)(x-3) = 0$  A1  
 $A(1, 1), B(3, 3)$  A1
- (b)  $\int_0^1 (x^3 - 4x^2 + 4x - x) \, dx = \left[ \frac{1}{4}x^4 - \frac{4}{3}x^3 + \frac{3}{2}x^2 \right]_0^1$  M1 M1  
 $= \frac{1}{4} - \frac{4}{3} + \frac{3}{2}$  dM1  
 $\frac{5}{12}$  A1
- (c) On  $1 < x < 3$ , curve – line =  $x(x-1)(x-3) < 0$ , so  $l$  is higher B1

