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A-LEVEL MATHEMATICS • FURTHER PURE 1

Transformations Using Matrices

Worksheet

Pearson Edexcel International A Level (WFM01) • Calculator allowed: show every step

Free Transformations Using Matrices worksheet for A-Level and International A-Level Maths, with full worked solutions

A **Examples** & Your turn page 2

B **Fluency** core skills page 4

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Tap a section to jump straight to it

By the end of this worksheet you can: find the matrix of a transformation • describe rotations, reflections, enlargements and stretches • combine transformations • use the area scale factor • *Stretch:* constants from images, powers of a rotation, and invariant points and lines.

Which questions are for me?

CORE The methods every A-level student must secure. Start here.

STRETCH Combined transformations in the right order, areas with negative determinants, and lines that map to themselves.

Every section has both, clearly labelled. Questions get harder as you go, and every skill in Sections A to D is needed in Section E.

Track your progress

Section	B	C	D	E
Core marks	23	10	13	13
Stretch marks	16	14	17	9
My score				

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A Worked examples & Your turn

Read each example carefully. Cover it up, then answer the question beside it using the same method.

CORE *Secure the method*

Worked example A1: columns are images

A linear transformation maps $(1,0)$ to $(3,1)$ and $(0,1)$ to $(-2,4)$. Write down its matrix.

The columns are the images of $\begin{pmatrix}1\\0\end{pmatrix}$ *and* $\begin{pmatrix}0\\1\end{pmatrix}$

$$\begin{pmatrix}3 & -2\\1 & 4\end{pmatrix}$$

Your turn A1

A linear transformation maps $(1,0)$ to $(0,2)$ and $(0,1)$ to $(5,-1)$. Write down its matrix.

Worked example A2: standard matrices

Write down the matrices of a reflection in the y-axis and a rotation of $90°$ anticlockwise about O

$$\begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$$

Your turn A2

Write down the matrices of a reflection in the x-axis and a rotation of $180°$ about O

Worked example A3: the area scale factor

A triangle of area 5 is transformed by $\begin{pmatrix}2 & 1\\0 & 3\end{pmatrix}$.

Find the area of its image.

Image area $= |\det| \times$ *area*

$\det = 6$

30

Your turn A3

A triangle of area 7 is transformed by $\begin{pmatrix}4 & -1\\2 & 1\end{pmatrix}$.

Find the area of its image.

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STRETCH

Worked example A4: describing a matrix

Describe fully the transformation $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$(1,0) \to (0,-1)$ *and* $(0,1) \to (1,0)$

rotation 90° clockwise about O

Your turn A4

Describe fully the transformation $\begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$

Worked example A5: one transformation followed by another

Find the matrix of a reflection in the x-axis followed by a stretch, scale factor 3, parallel to the y-axis.

$\mathbf{P}$ *followed by* $\mathbf{Q}$ *is* $\mathbf{QP}$

$$\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}$$

Your turn A5

Find the matrix of a rotation of 90° anticlockwise about O followed by an enlargement, scale factor 2, centre O

Worked example A6: exact rotations

Write down the matrix of a rotation of 30° anticlockwise about O, in exact form.

$$\begin{pmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{pmatrix}$$

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

Your turn A6

Write down the matrix of a rotation of 120° anticlockwise about O, in exact form.

Worked example A7: invariant points

Find the points that are mapped to themselves by

$$\begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}$$

$3x - 2y = x$ and $x = y$

every point on $y = x$

Your turn A7

Find the points that are mapped to themselves by

$$\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$$

B Fluency

Each question is a designed sequence. From one part to the next, ask yourself: what changed, and what did it do to the answer?

CORE *Secure the method*

B1 Write down the matrix of

(a) a reflection in $y = x$

(b) a reflection in $y = -x$

(c) an enlargement, scale factor 4, centre O

(d) a stretch, scale factor 3, parallel to the x-axis

[4 marks]

B2 Write down, in exact form, the matrix of a rotation about O of

(a) 90° clockwise

(b) 60° anticlockwise

(c) 135° anticlockwise

(d) 330° anticlockwise

[4 marks]

B3 Describe fully the transformation represented by

(a) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

(d) $\begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$

[4 marks]

B4 The triangle T has vertices $(0,0)$, $(2,0)$ and $(0,1)$, and $\mathbf{M} = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$

(a) Find the vertices of the image of T

(b) Find the area of the image directly.

(c) Find $\det \mathbf{M}$ and compare.

(d) A shape of area 4 is transformed by $\mathbf{M}$. Find its image's area.

[4 marks]

B5 $\mathbf{P}$ is a stretch, scale factor 2, parallel to the x-axis, and $\mathbf{Q}$ is a rotation of 90° anticlockwise about O

(a) Find the matrix for $\mathbf{P}$ followed by $\mathbf{Q}$

(b) Find the matrix for $\mathbf{Q}$ followed by $\mathbf{P}$

(c) Show that both have the same area scale factor, and explain why. **[3 marks]**

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B6 $\mathbf{M} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

(a) Find $\mathbf{M}^{-1}$

(b) Find the point mapped by $\mathbf{M}$ to $(5, 3)$

(c) Find the image of the line $y = x$

(d) State the area scale factor.

[4 marks]

STRETCH

B7 $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ maps $(1, 2)$ to $(4, 7)$ and $(3, 1)$ to $(7, 6)$

(a) Find a and b

(b) Find c and d

(c) Find $\det \mathbf{M}$

(d) Find the area of the image of a square of side 2

[4 marks]

B8 $\mathbf{M} = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}$

(a) Show that $(1, 1)$ is mapped to itself.

(b) Find the line of points mapped to themselves.

(c) Find the image of $(1, 0)$

(d) Find the area scale factor.

[4 marks]

B9 $\mathbf{A}$ is a reflection in $y = x$ and $\mathbf{B}$ is a rotation of 90° anticlockwise about O

(a) Find $\mathbf{AB}$

(b) Find $\mathbf{BA}$

(c) Describe the transformation $\mathbf{AB}$

(d) Describe the transformation $\mathbf{BA}$

[4 marks]

B10 $\mathbf{M} = \begin{pmatrix} k & 3 \\ 2 & k \end{pmatrix}$

(a) Find $\det \mathbf{M}$ in terms of k

(b) Find k when the area scale factor is 10

(c) Find k when $\mathbf{M}$ maps every shape to zero area.

(d) When $k = 4$, a triangle has area 3. Find the area of its image.

[4 marks]

C Reasoning

Show your working and explain your thinking. A correct answer without a reason does not score full marks.

CORE *Secure the method*

C1 Amir says that the transformation $\mathbf{P}$ followed by the transformation $\mathbf{Q}$ is represented by $\mathbf{PQ}$

Explain his error. **[2 marks]**

C2 A shape of area 12 is transformed by a matrix with determinant 3. Bea says the image has area 4

Explain her error, and find the area. **[2 marks]**

C3 Cal says that $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is a reflection in the line $y = -x$

Explain his error, and describe the transformation fully. **[3 marks]**

C4 Dev describes $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ as "an enlargement, scale factor 2".

Explain why this is not a full and correct description, and give one. **[3 marks]**

STRETCH

C5 A shape of area 4 is transformed by a matrix with determinant -3. Eve says the image has area -12

Explain her error, and state what the negative determinant tells you. **[3 marks]**

C6 Fay says that $\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ is a rotation of 30° anticlockwise about O

Explain her error, and describe the transformation. **[3 marks]**

C7 $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{CA} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$. Gus finds $\mathbf{C} = \mathbf{A}^{-1}\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$

Explain his error, and find $\mathbf{C}$ **[3 marks]**

C8 $\mathbf{R}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ represents a rotation through θ anticlockwise about O

(a) Show that $\mathbf{R}(\alpha)\mathbf{R}(\beta) = \mathbf{R}(\alpha + \beta)$

(3)

(b) Hence show that $\mathbf{R}(\theta)^{-1} = \mathbf{R}(-\theta)$, and that a rotation does not change area.

(2)

[5 marks]

D Problem solving

You decide the method. Read the whole question first, and show every step.

CORE *Secure the method*

D1 The matrix $\mathbf{P} = \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}$ represents the linear transformation P of the plane.

(a) Describe fully the transformation P as a single geometrical transformation.

(1)

The transformation Q is a reflection in the x-axis.

(b) Write down the matrix $\mathbf{Q}$ representing Q

(1)

The transformation Q followed by the transformation P is represented by a single matrix, and it maps the point with coordinates $(3, -2)$ onto the point A

(c) Determine the coordinates of the point A, showing clearly the order in which you multiply the matrices.

(1)

[3 marks]

D2 The matrix $\mathbf{M} = \begin{pmatrix} 3 & k \\ k & 2 \end{pmatrix}$, where k is a positive constant, represents the linear transformation T of the plane.

Given that the transformation T maps the point with coordinates $(1, 2)$ onto the point with coordinates $(11, 8)$,

(a) determine the value of k

(2)

A quadrilateral Q is transformed onto the quadrilateral Q' by the matrix $\mathbf{M}$, and the area of Q' is 60 square units.

(b) Using the value of k found in part (a), determine the area of the original quadrilateral Q, explaining how you use the determinant of $\mathbf{M}$

(2)

[4 marks]

D3 The matrix $\mathbf{B} = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$, where p and q are integers, represents a linear transformation of the plane.

(a) State the values of p and q when $\mathbf{B}$ represents an enlargement about the origin with scale factor 3, and when it represents a reflection in the x-axis.

(2)

(b) Given instead that the transformation represented by $\mathbf{B}$ maps the point with coordinates $(2, 5)$ onto the point with coordinates $(-6, 10)$, determine the values of p and q

(1)

[3 marks]

D4 The matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ represents a stretch, scale factor 2, parallel to the y-axis. The matrix $\mathbf{B}$ represents a rotation of 150° anticlockwise about the origin. The transformation represented by $\mathbf{A}$ followed by the transformation represented by $\mathbf{B}$ is represented by the matrix $\mathbf{C}$

(a) Determine the matrix $\mathbf{C}$, giving each element in exact form, and showing the order in which you multiply.

(2)

(b) A shape S of area 3 square units is transformed onto the shape S' by the matrix $\mathbf{C}$. Find the area of S', giving a reason for your answer.

(1)

[3 marks]

STRETCH

D5 The linear transformation W of the plane is represented by the 2×2 matrix $\mathbf{W} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$

(a) Show that there is a real number λ for which W maps the point $(\lambda, 1)$ onto the point $(4\lambda, 4)$, giving the exact value of λ

(3)

(b) Hence write down the equation of a line through the origin with the property that every point on the line is mapped by W onto a point on the same line, and state the factor by which distances from the origin along this line are multiplied.

(1)

[4 marks]

D6 The matrix $\mathbf{A}$ is given by $\mathbf{A} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$, and $\mathbf{I}$ is the 2×2 identity matrix.

(a) Determine the matrix $\mathbf{A}^2$ and describe fully the single geometrical transformation it represents.

(2)

(b) Hence determine the smallest positive integer value of n for which $\mathbf{A}^n = \mathbf{I}$

(1)

(c) Describe fully the transformation represented by $\mathbf{A}^4$

(1)

[4 marks]

D7 The matrix $\mathbf{A} = \begin{pmatrix} k & 2 \\ 5 & k \end{pmatrix}$, where k is a constant, represents a transformation that maps a triangle T onto a triangle T'. The area of the triangle T' is four times the area of the triangle T

Using the determinant of $\mathbf{A}$ as an area scale factor, and remembering that the determinant may be negative, determine all the possible values of k, giving your answers in exact form. **[4 marks]**

D8 The matrices $\mathbf{M}$ and $\mathbf{N}$ are given by $\mathbf{M} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ and $\mathbf{N} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^3$. The transformation represented by the matrix $\mathbf{M}$ followed by the transformation represented by the matrix $\mathbf{N}$ is represented by the matrix $\mathbf{B}$

(a) Determine $\mathbf{N}$ in the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c and d are integers.

(1)

(b) Determine $\mathbf{B}$

(2)

(c) A hexagon S is transformed onto the hexagon S' by $\mathbf{B}$. Given that the area of S' is 72 square units, determine the area of S

(2)

[5 marks]

E Exam practice

Exam-style questions in the style of WFM01. Write your answers in the spaces, and show every step.

CORE *Exam questions for everyone*

E1 The 2×2 matrix $\mathbf{P} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ represents the linear transformation P of the plane.

(a) Describe P fully as a single geometrical transformation.

(2)

The transformation Q is an enlargement about the origin with scale factor 3

(b) Write down the matrix $\mathbf{Q}$ representing Q

(1)

(c) The transformation P followed by the transformation Q maps the point $(2, 5)$ to the point B. Determine the coordinates of B

(2)

(d) State the area scale factor of the transformation P followed by Q, giving a reason for your answer, and explain what the sign of the determinant of the combined matrix tells you.

(1)

(Total for Question E1 is 6 marks)

E2 The 2×2 matrix $\mathbf{A} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ represents the linear transformation U of the plane.

(a) Describe fully the single geometrical transformation U

(2)

The transformation V, represented by the 2×2 matrix $\mathbf{B}$, is a reflection in the y-axis.

(b) Write down the matrix $\mathbf{B}$

(1)

The transformation U followed by the transformation V is represented by the matrix $\mathbf{C}$

(c) Determine the matrix $\mathbf{C}$

(2)

The transformation W is represented by the matrix $2\mathbf{C}$. The transformation W' maps the image of any point under W back to the original point.

(d) Determine the matrix that represents W'

(2)

(Total for Question E2 is 7 marks)

STRETCH

E3 The linear transformation T of the plane is represented by the 2×2 matrix $\mathbf{M} = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$

(a) A triangle of area 5 square units is transformed by T. Find the area of its image.

(1)

A line through the origin is called invariant under T if every point of the line is mapped onto a point of the same line.

(b) Show that the lines $y = mx$ that are invariant under T satisfy $m^2 - 3m + 2 = 0$, and hence find their equations.

(4)

(c) Show that every point on one of these lines is mapped to a point three times as far from the origin.

(1)

(d) The matrix $\mathbf{N}$ is such that $\mathbf{NM} = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$. Determine $\mathbf{N}$

(3)

(Total for Question E3 is 9 marks)

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