



eClassroom

A-LEVEL MATHEMATICS • PURE 2

# Proof

## Worked Solutions

Pearson Edexcel International A Level (WMA12) & A Level (9MA0) • Calculator allowed: show every step

Full worked solutions for the free Proof worksheet, A-Level and International A-Level Maths



Tap a section to jump straight to it

**Full worked solutions** for every question on the Pure 2 Proof worksheet, in the same order and with the same question numbers.

**How to mark your work.** Tick each step you got right, not just the final answer. At A-level most marks are for method, so a wrong final answer can still earn most of the marks. If you lost marks, find the *first* line where your working is different from ours. That is the thing to fix.

In Section E: **M** = method mark, **dM** = method mark that depends on the previous M mark, **A** = accuracy mark (needs the M mark before it), **B** = independent mark.

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**A****Your turn answers**

Compare each line of your working with the solution.

**CORE**

**A1**  $2n(2n+2) = 4n(n+1)$ . One of  $n$  and  $n+1$  is even, so  $n(n+1) = 2m$  for an integer  $m$ , and  $4n(n+1) = 8m$

a multiple of 8, as required

**A2**  $n = 3k$ :  $n^2 = 9k^2 = 3(3k^2)$

$n = 3k+1$ :  $n^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$

$n = 3k+2$ :  $n^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$

every integer is one of these, so  $n^2$  is a multiple of 3 or one more than a multiple of 3

**A3**  $n = 3$ :  $2^3 + 1 = 9 = 3 \times 3$ , which is not prime:  $n = 3$

**STRETCH**

**A4**  $n^2 - 8n + 20 = (n-4)^2 + 4$ , and  $(n-4)^2 \geq 0$ , so  $n^2 - 8n + 20 \geq 4 > 0$

**A5**  $pq > 0$  because  $p$  and  $q$  are positive, so multiplying by  $pq$  keeps the inequality:  $p^2 > q^2$   
 $(p-q)(p+q) > 0$ , and  $p+q > 0$ , so  $p-q > 0$ :  $p > q$

**A6**  $\sqrt{37} \approx 6.1$ , so test 2, 3 and 5: 37 is odd,  $37 = 3 \times 12 + 1$ , and 37 does not end in 0 or 5

no prime factor up to  $\sqrt{37}$ , so 37 is prime

**A7**  $a = 2, b = -1$ :  $2 > -1$ , but  $\frac{1}{2} > -1$ , so  $\frac{1}{a} < \frac{1}{b}$  fails:  $a = 2, b = -1$



**B****Fluency solutions**

One mark for each correct part. The notes in *italics* explain why each sequence was chosen.

**CORE**

**B1** (a)  $2n$  (b)  $2n+1$  (c)  $n, n+1, n+2$  (d)  $3n+1$

*Every proof starts by writing the numbers in general, never with examples.*

**B2** (a)  $(2m+1) + (2n+1) = 2(m+n+1)$ : even (b)  $(2m+1)(2n+1) = 2(2mn+m+n) + 1$ : odd  
 (c)  $n + (n+1) + (n+2) = 3(n+1)$ : a multiple of 3 (d)  $(2n+1)^2 - 1 = 4n^2 + 4n = 4(n^2 + n)$ : a multiple of 4

*Two different odd numbers need two different letters,  $m$  and  $n$*

**B3** (a) 2 is prime and even. (b)  $x = \frac{1}{2}$ :  $\frac{1}{4} < \frac{1}{2}$  (c)  $a = b = 1$ :  $\sqrt{2} \neq 2$

(d)  $n = 11$ :  $121 - 11 + 11 = 121 = 11 \times 11$

*One counter-example is enough, and it must be shown to work.*

**B4** (a)  $(x+2)^2 + 1 > 0$  (b)  $(x-5)^2 + 1 > 0$  (c)  $2(x^2 - 2x) + 3$ :  $2(x-1)^2 + 1 > 0$

(d)  $-(x^2 - 6x + 10)$ :  $-((x-3)^2 + 1) < 0$

*A square is at least 0; the constant left over decides the sign.*

**B5** (a)  $n^2 + 3$  gives 4, 7, 12, 19, 28: none is a multiple of 5

(b)  $p = 5, 7, 11, 13, 17, 19$  give 24, 48, 120, 168, 288, 360: all are multiples of 24

(c)  $3^n + 1$  gives 4, 10, 28, 82: all even

*A table of every case, then a one-line conclusion: no extra rows, no missing ones.*

**B6** (a) false:  $2 + 3 = 5$  (b) true:  $n(n+1)$  has one even factor

(c) true:  $(2n+1)^2 = 2(2n^2 + 2n) + 1$  (d) false:  $x = \frac{1}{2}$  gives  $\frac{1}{8} < \frac{1}{2}$

*Decide first whether to look for a proof or a counter-example.*



## STRETCH

- B7** (a)  $n = 2k$ :  $4k^2$ ;  $n = 2k + 1$ :  $4(k^2 + k) + 1$ : **a multiple of 4 or one more**  
 (b)  $n = 3k$ :  $3(3k^2) + 1$ ;  $3k + 1$ :  $3(3k^2 + 2k) + 2$ ;  $3k + 2$ :  $3(3k^2 + 4k + 1) + 2$ : **never a multiple of 3**  
 (c)  $n = 3k$ :  $3(9k^3 + 2k)$ ;  $3k + 1$ :  $3(9k^3 + 9k^2 + 5k + 1)$ ;  $3k + 2$ :  $3(9k^3 + 18k^2 + 14k + 4)$ : **a multiple of 3**  
 (d)  $n(n+1)$  is even, and one of  $n, n+1, 2n+1$  is a multiple of 3 (check  $n = 3k, 3k+1, 3k+2$ ): **a multiple of 6**  
 Choose cases by the remainder on dividing by the number in the question.

- B8** (a)  $(a-b)^2 \geq 0$ , so  $a^2 - 2ab + b^2 \geq 0$ :  **$a^2 + b^2 \geq 2ab$**   
 (b) divide (a) by  $ab > 0$ :  **$\frac{a}{b} + \frac{b}{a} \geq 2$**  (c)  $(\sqrt{a} - \sqrt{b})^2 \geq 0$ , so  $a + b \geq 2\sqrt{ab}$ :  **$\frac{a+b}{2} \geq \sqrt{ab}$**   
 (d)  $(x-1)^2 \geq 0$ , so  $x^2 + 1 \geq 2x$ ; divide by  $x > 0$ :  **$x + \frac{1}{x} \geq 2$**   
 Each starts from a square, and every division is by something positive.

- B9** (a)  **$x = -2$**  (b)  **$n = 1$** :  $1 + 1 = 2$  (c)  **$p = \sqrt{2}, q = -\sqrt{2}$** :  $p + q = 0$  (d)  **$x = 3$** :  $8 < 9$   
 The condition must be true, and the conclusion false.

- B10** (a)  $\sqrt{41} \approx 6.4$ : 2, 3, 5 do not divide 41: **41 is prime** (b)  $\sqrt{97} \approx 9.8$ : 2, 3, 5, 7 do not divide 97:  
**97 is prime**  
 (c)  **$91 = 7 \times 13$**   
 (d) If  $N = ab$  with both  $a$  and  $b$  greater than  $\sqrt{N}$ , then  $ab > N$ ; so  
**any factor pair contains a number up to  $\sqrt{N}$**   
 Only primes need testing, because any factor has a prime factor that is at least as small.

## C

## Reasoning solutions

Your wording can be different. Check that your reason makes the same point.

## CORE

- C1** Examples cannot cover every case. A proof must work for all odd numbers:  
 **$(2m+1) + (2n+1) = 2(m+n+1)$ , which is even**
- C2**  $2n+1$  and  $2n+3$  are consecutive odd numbers, so her proof only covers that special case.  
**use  $2m+1$  and  $2n+1$ , with different letters**:  $(2m+1)(2n+1) = 2(2mn+m+n) + 1$ , odd



**C3** Kai has only checked even  $n$ , and has not written a conclusion.

$$n = 2k + 1: 4k^2 + 6k + 3 = 2(2k^2 + 3k + 1) + 1, \text{ odd}$$

every integer is even or odd, so  $n^2 + n + 1$  is always odd

**C4** Jo's values do not satisfy the condition  $x > y$ , so they cannot disprove the statement.

$$x = 1, y = -1: 1 > -1, \text{ but } \frac{1}{1} > \frac{1}{-1}, \text{ so the conclusion fails}$$

### STRETCH

**C5** Max must say why dividing keeps the inequality:  $ab > 0$ . To go from  $a^2 > b^2$  to  $a > b$  he needs  $a + b > 0$

$$a^3b - ab^3 = ab(a - b)(a + b) > 0; ab > 0 \text{ and } a + b > 0 \text{ since } a \text{ and } b \text{ are positive, so } a - b > 0, \text{ so } a > b$$

**C6**  $2^2 - 1 = 3$ ,  $2^3 - 1 = 7$ ,  $2^5 - 1 = 31$ ,  $2^7 - 1 = 127$ , all prime.

$$11 \text{ is prime, but } 2047 = 23 \times 89: \text{ the statement is false, } p = 11$$

**C7** Both sides are positive, so squaring keeps the inequality:  $(\sqrt{3} + \sqrt{5})^2 = 8 + 2\sqrt{15}$

$$8 + 2\sqrt{15} < 16 \text{ exactly when } \sqrt{15} < 4, \text{ that is } 15 < 16, \text{ which is true, so } \sqrt{3} + \sqrt{5} < 4$$

**C8** (a)  $n^3 - n = (n - 1)n(n + 1)$ , three consecutive integers: one is even and one is a multiple of 3:

a multiple of 6

(b)  $n = 2$ :  $8 - 2 = 6$ , not a multiple of 12

(c)  $n$  odd:  $n - 1$  and  $n + 1$  are consecutive even numbers, so one is a multiple of 4: their product is a multiple of 8. With the factor 3 from (a):

a multiple of 24



## D

## Problem solving solutions

There is often more than one correct method. Any complete, correct method scores full marks.

## CORE

**D1**  $n = 3k$ :  $n^2 = 3(3k^2)$ , a multiple of 3

$$n = 3k + 1: n^2 = 3(3k^2 + 2k) + 1$$

$$n = 3k + 2: n^2 = 3(3k^2 + 4k + 1) + 1$$

every integer is one of these forms, so the result holds for all integers

**D2** (a)  $n = 16$ :  $256 + 16 + 17 = 289 = 17^2$ , not prime

(b)  $n = 2k$ :  $4k^2 + 2k + 17 = 2(2k^2 + k + 8) + 1$ , odd

$$n = 2k + 1: 4k^2 + 6k + 19 = 2(2k^2 + 3k + 9) + 1, \text{ odd}$$

every integer is even or odd, so  $n^2 + n + 17$  is always odd

**D3**  $ab > 0$  because  $a$  and  $b$  are positive, so multiply both sides by  $ab$ :  $a^2 + b^2 = 2ab$

$$a^2 - 2ab + b^2 = 0, \text{ so } (a - b)^2 = 0, \text{ so } a - b = 0: a = b$$

**D4**  $x^2 - 4x + 7 = (x - 2)^2 + 3$ , and  $(x - 2)^2 \geq 0$ , so  $x^2 - 4x + 7 \geq 3$ ; equality when  $(x - 2)^2 = 0$ :  $x = 2$

## STRETCH

**D5** (a)  $x^3 - y^3 - x^2y + xy^2 = x^2(x - y) + y^2(x - y) = (x - y)(x^2 + y^2) < 0$

$$x^2 + y^2 > 0 \text{ because } x \text{ and } y \text{ are positive, so dividing keeps the inequality: } x - y < 0: x < y$$

(b) **Yes**: if the inequality holds,  $x$  and  $y$  are not both 0, so  $x^2 + y^2 > 0$  for any real  $x$  and  $y$ , and the same steps work.

**D6** (a)  $(2k + 1)^2 = 4k^2 + 4k + 1 = 4k(k + 1) + 1$ . One of  $k, k + 1$  is even, so  $k(k + 1) = 2m$ :  $(2k + 1)^2 = 8m + 1$

(b)  $(8m + 1) - (8p + 1) = 8(m - p)$ : a multiple of 8

**D7** (a)  $14^2 = 196 < 221 < 15^2 = 225$ : if  $221 = ab$ , one of  $a$  and  $b$  is at most  $\sqrt{221} < 15$ , so a prime factor up to 14 would show

(b)  $221 = 13 \times 17$

(c)  $\sqrt{223} < 15$ : test 2, 3, 5, 7, 11 and 13; none divides 223 (for example  $223 = 13 \times 17 + 2$ ): 223 is prime



- D8** (a)  $n(n+3) = n^2 + 3n$  and  $(n+1)(n+2) = n^2 + 3n + 2$ . With  $u = n^2 + 3n$ :  $u(u+2) + 1 = u^2 + 2u + 1 = (u+1)^2$   
 so  $n(n+1)(n+2)(n+3) + 1 = (n^2 + 3n + 1)^2$  **as required**
- (b)  $n = 5$ :  $25 + 15 + 1 = 41$ :  $41^2$  (= 1681)
- (c)  $1 \times 2 \times 3 + 1 = 7$ , which is not a square number.

**E****Exam practice solutions**

Mark-scheme notes are shown on the right of each line.

**CORE**

Exam questions for everyone

- E1** (i)  $n = 4$ :  $16 + 4 + 1 = 21 = 3 \times 7$  *M1: a value tested*  
**21 is not prime, so the statement is false** *A1: with a conclusion*
- (ii)  $n = 2k$ :  $8k^3 + 2k = 2(4k^3 + k)$ , even *M1: one case*  
 $n = 2k + 1$ :  $8k^3 + 12k^2 + 8k + 2 = 2(4k^3 + 6k^2 + 4k + 1)$ , even *M1: both cases*  
**every integer is even or odd, so  $n^3 + n$  is always even** *A1: both correct and a conclusion*
- E2** (a)  $x(y+1) - y(x+1) = xy + x - xy - y = x - y$  *M1: compares the cross-products*  
 $x > y$ , so  $x(y+1) - y(x+1) > 0$  *A1*  
 $y(y+1) > 0$  because  $y$  is positive, so dividing by  $y(y+1)$  keeps the inequality *M1: sign stated*  
 $\frac{x}{y} > \frac{x+1}{y+1}$  *A1\**
- (b)  $x = 1, y = -\frac{1}{2}$ :  $x > y$ , but  $\frac{x}{y} = -2$  and  $\frac{x+1}{y+1} = 4$  *M1: the condition holds*  
 **$-2 < 4$ , so the result does not hold** *A1*



## STRETCH

- E3** (a) The last digit of  $n^2$  depends only on the last digit of  $n$  B1  
last digit of  $n$ : 0, 1, 2, 3, 4, 5, 6, 7, 8, 9  
last digit of  $n^2$ : 0, 1, 4, 9, 6, 5, 6, 9, 4, 1 M1 A1: all ten cases  
**always 0, 1, 4, 5, 6 or 9** A1: conclusion
- (b)  $10m + 3$  ends in 3, which is not in the list: **no such  $n$**  B1
- (c) **26** ends in 6 but is not a square ( $5^2 = 25 < 26 < 36 = 6^2$ ) B1
- (d)  $n^4 = (n^2)^2$ : squaring last digits 0, 1, 4, 5, 6, 9 gives 0, 1, 6, 5, 6, 1 M1  
**always 0, 1, 5 or 6** A1

