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INTERNATIONAL A LEVEL MATHEMATICS • FURTHER PURE 2 (WFM02)

Further Pure 2 Practice Paper 2

Worked Solutions

AIMING FOR A

Written in the style of the Pearson Edexcel International Advanced Level paper (WFM02/01) • 75 marks

Every question solved in full on the paper itself, the way your teacher would write it out

How to read these solutions. Each page is the real question page, with a full answer written on the answer lines.

Red pen

M1 A1

boxed

the working a full-marks answer shows, with a short note on why each step is taken

the mark earned on that line: M = method, A = accuracy, B = independent mark

the final answer to each part

Sit the paper first. These solutions are most useful after you have tried every question yourself under timed conditions. Mark your paper with the mark scheme, then use this booklet to see exactly how each lost mark could have been earned.

Q	Topic	Marks	Page
1	A modulus inequality and a sketch	7	2
2	A differential equation by substitution	8	3
3	Differences with three partial fractions	9	4
4	A line mapped to a circle	9	5
5	A Taylor series from a differential equation	10	6
6	Fourth roots and the square they make	10	7
7	A particular integral inside the complementary function	11	8
8	The area inside two polar curves	11	9

Question paper & mark scheme | Exam questions by topic | More practice papers

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1. **(a)** On the same axes, sketch the graphs of $y = |x^2 - 4|$ and $y = 3x$, showing the coordinates of any points where the graph of $y = |x^2 - 4|$ meets the coordinate axes.

(2)

(b) Find the values of x for which $|x^2 - 4| = 3x$

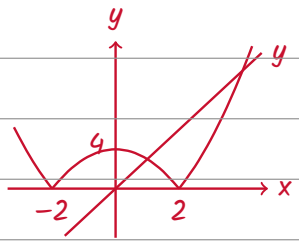
(2)

(c) Hence, or otherwise, find the set of values of x for which

$$|x^2 - 4| > 3x$$

(3)

(a) $y = |x^2 - 4|$: reflect the part below the x-axis; it touches $(\pm 2, 0)$ and meets $(0, 4)$ B1



B1

(b) $x^2 - 4 = 3x \Rightarrow (x - 4)(x + 1) = 0$: $x = 4$ ($x = -1$ would make $3x < 0$, so reject) M1

$4 - x^2 = 3x \Rightarrow (x + 4)(x - 1) = 0$: $x = 1$ (reject $x = -4$). $\boxed{x = 1,\ x = 4}$ A1

(c) The curve is above the line except between $x = 1$ and $x = 4$. For $x < 0$, $|x^2 - 4| \geqslant 0 > 3x$ M1

$\boxed{x < 1}$ or $\boxed{x > 4}$ A1 A1

(Total for Question 1 is 7 marks)

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2.

$$x\frac{dy}{dx} + y = x^3y^2, \qquad x > 0$$

(a) Show that the substitution $z = \frac{1}{y}$ transforms this differential equation into the equation

$$\frac{dz}{dx} - \frac{z}{x} = -x^2$$

(3)

(b) By solving the equation in part (a), find the particular solution of the original differential equation for which $y = 1$ when $x = 1$, giving your answer in the form $y = f(x)$

(5)

(a) $y = z^{-1}$, so $\frac{dy}{dx} = -z^{-2}\frac{dz}{dx}$ B1

Substitute: $x(-z^{-2}\frac{dz}{dx}) + z^{-1} = x^3 z^{-2}$ M1

$\times(-z^2)$: $\frac{dz}{dx} - \frac{z}{x} = -x^2$ as required A1

(b) Integrating factor $= e^{-\int 1/x\,dx} = e^{-\ln x} = \frac{1}{x}$ B1

$\frac{d}{dx}(\frac{z}{x}) = -x \Rightarrow \frac{z}{x} = -\frac{1}{2}x^2 + c \Rightarrow z = cx - \frac{1}{2}x^3$ M1 A1

$y = \frac{1}{z} = \frac{1}{cx - x^3/2}$. $x = 1, y = 1$: $1 = \frac{1}{c - 1/2}$, so $c = \frac{3}{2}$ M1

$y = \frac{1}{(3x - x^3)/2}$: $y = \frac{2}{3x - x^3}$ A1

(Total for Question 2 is 8 marks)

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3. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

(a) Express $\dfrac{4}{r(r+2)(r+4)}$ in partial fractions.

(3)

(b) Hence, using the method of differences, find

$$\sum_{r=1}^{n} \frac{4}{r(r+2)(r+4)}$$

giving your answer in the form $a - \dfrac{1}{2}\left(\dfrac{1}{n+1} + \dfrac{1}{n+2} - \dfrac{1}{n+3} - \dfrac{1}{n+4}\right)$, where a is a rational number to be found.

(4)

(c) Hence find the limit of $\displaystyle\sum_{r=1}^{n} \frac{4}{r(r+2)(r+4)}$ as $n \to \infty$

(2)

(a) $4 = A(r+2)(r+4) + Br(r+4) + Cr(r+2)$; put $r = 0, -2, -4$ M1

$8A = 4$, $-4B = 4$, $8C = 4$: $A = \frac{1}{2}$, $B = -1$, $C = \frac{1}{2}$ A1

$1/(2r) - 1/(r+2) + 1/(2(r+4))$ A1

(b) Split: $\frac{1}{2}[(\frac{1}{r} - \frac{1}{r+2}) - (\frac{1}{r+2} - \frac{1}{r+4})]$ (each bracket cancels in steps of 2) M1

$\sum (\frac{1}{r} - \frac{1}{r+2}) = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$; $\sum (\frac{1}{r+2} - \frac{1}{r+4}) = \frac{1}{3} + \frac{1}{4} - \frac{1}{n+3} - \frac{1}{n+4}$ A1

Constant part: $\frac{1}{2}(\frac{3}{2} - \frac{7}{12}) = \frac{1}{2} \times \frac{11}{12} = \frac{11}{24}$ M1

$\Sigma = \frac{11}{24} - \frac{1}{2}(\frac{1}{n+1} + \frac{1}{n+2} - \frac{1}{n+3} - \frac{1}{n+4})$, $a = 11/24$ A1

(c) As $n \to \infty$, each $\frac{1}{n+k} \to 0$ M1

$11/24$ A1

(Total for Question 3 is 9 marks)

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4. The transformation T from the z-plane, where $z = x + \mathrm{i}y$, to the w-plane, where $w = u + \mathrm{i}v$, is given by

$$w = \frac{z + 2\mathrm{i}}{z - \mathrm{i}}, \qquad z \neq \mathrm{i}$$

(a) Show that the real axis in the z-plane is mapped by T onto a circle in the w-plane, and find the centre and radius of this circle.

(6)

(b) On an Argand diagram, shade the region of the w-plane onto which T maps the region of the z-plane for which $y > 0$

(3)

(a) $w(z - i) = z + 2i \Rightarrow z(w - 1) = iw + 2i \Rightarrow z = \frac{i(w + 2)}{w - 1}$ M1

$w = u + iv$: $z = \frac{i(u + 2 + iv)(u - 1 - iv)}{(u - 1)^2 + v^2}$ (× the conjugate of the bottom) M1

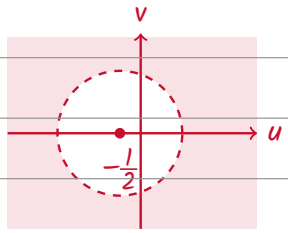
$(u + 2 + iv)(u - 1 - iv) = (u^2 + u - 2 + v^2) - 3iv$, so $z = \frac{3v + i(u^2 + u - 2 + v^2)}{(u - 1)^2 + v^2}$ A1

z on the real axis: $\mathrm{Im}\, z = 0 \Rightarrow u^2 + u - 2 + v^2 = 0$ M1

$(u + \frac{1}{2})^2 + v^2 = \frac{9}{4}$ A1

a circle, centre $-\frac{1}{2}$ $(-\frac{1}{2}, 0)$, radius $\frac{3}{2}$ ($w = 1$ is on it but is not reached) A1

(b) $y > 0$: $\mathrm{Im}\, z > 0$, so $u^2 + u - 2 + v^2 > 0$, outside the circle (check: $z = 2i \rightarrow w = 4$) M1 A1



B1

(Total for Question 4 is 9 marks)

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5.

$$\frac{dy}{dx} = y^2 - x$$

Given that $y = 2$ when $x = 1$

(a) find the value of $\frac{d^2y}{dx^2}$ and the value of $\frac{d^3y}{dx^3}$ when $x = 1$

(3)

(b) find a series solution for y in ascending powers of $(x - 1)$, up to and including the term in $(x - 1)^3$

(4)

(c) use your series to estimate the value of y when $x = 1.1$, giving your answer to 3 decimal places.

(2)

(d) Explain why your series is unlikely to give a reliable estimate of the value of y when $x = 2$

(1)

(a) $\frac{d^2y}{dx^2} = 2y\frac{dy}{dx} - 1$, $\frac{d^3y}{dx^3} = 2(\frac{dy}{dx})^2 + 2y\frac{d^2y}{dx^2}$ (product rule) M1

$x = 1, y = 2$: $\frac{dy}{dx} = 4 - 1 = 3$, $\frac{d^2y}{dx^2} = 2(2)(3) - 1 = \boxed{11}$ A1

$\frac{d^3y}{dx^3} = 2(3)^2 + 2(2)(11) = 18 + 44 = \boxed{62}$ A1

(b) $y = y(1) + y'(1)(x - 1) + \frac{y''(1)}{2!}(x - 1)^2 + \frac{y'''(1)}{3!}(x - 1)^3$ M1

$= 2 + 3(x - 1) + \frac{11}{2}(x - 1)^2 + \frac{62}{6}(x - 1)^3$ A1 A1

$\boxed{y \approx 2 + 3(x - 1) + \frac{11}{2}(x - 1)^2 + \frac{31}{3}(x - 1)^3}$ A1

(c) $x - 1 = 0.1$: $2 + 0.3 + 0.055 + 0.01033... = 2.3653...$ M1

$\boxed{2.365}$ A1

(d) $x = 2$ is far from 1, so $(x - 1) = 1$ is not small and the missing terms are not small B1

(Total for Question 5 is 10 marks)

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6. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

$$w = -8 - 8\sqrt{3}\mathrm{i}$$

(a) Express w in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$ and $-\pi < \theta \leqslant \pi$

(2)

(b) Solve the equation $z^4 = w$, giving your answers in the form $a + b\mathrm{i}$, where a and b are exact real numbers.

(5)

(c) The points representing the roots of $z^4 = w$ are the vertices of a quadrilateral on an Argand diagram. Show these points on an Argand diagram and find the exact area of the quadrilateral.

(3)

(a) $r = \sqrt{8^2 + (8\sqrt{3})^2} = \sqrt{256} = 16$; w is in the 3rd quadrant: $\theta = -\pi + \frac{\pi}{3} = -\frac{2\pi}{3}$ M1

$w = 16(\cos(-\frac{2\pi}{3}) + i\sin(-\frac{2\pi}{3}))$ A1

(b) $|z| = 16^{1/4} = 2$ B1

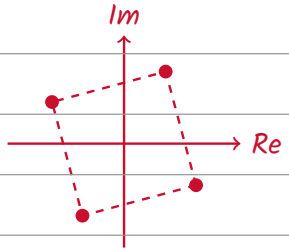
$\arg z = \frac{1}{4}(-\frac{2\pi}{3} + 2k\pi) = -\frac{\pi}{6} + \frac{k\pi}{2}$ M1

$k = 0, 1, 2, -1$: $\arg z = -\frac{\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{6}, -\frac{2\pi}{3}$ M1

$2(\cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6})) = \sqrt{3} - i$, $2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) = 1 + \sqrt{3}i$ A1

$\sqrt{3} - i,\ 1 + \sqrt{3}i,\ -\sqrt{3} + i,\ -1 - \sqrt{3}i$ A1

(c) The roots are 90° apart on a circle of radius 2, so they form a square:



B1

Diagonals = 2 × 2 = 4: area = $\frac{1}{2} \times 4 \times 4 = 8$ (or side $2\sqrt{2}$, $(2\sqrt{2})^2 = 8$) M1 A1

(Total for Question 6 is 10 marks)

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7.

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 10e^{3x}$$

(a) Explain why a particular integral of the form $y = \lambda e^{3x}$ cannot be used, and find a suitable particular integral.

(5)

(b) Find the general solution of the differential equation.

(2)

(c) Find the particular solution for which $y = 0$ and $\frac{dy}{dx} = 1$ when $x = 0$

(4)

(a) Auxiliary: $m^2 - m - 6 = 0 \Rightarrow (m - 3)(m + 2) = 0$, $m = 3$ or -2

e^{3x} is already in the CF, so λe^{3x} makes the left side 0, not $10e^{3x}$ B1

Try $y = \lambda x e^{3x}$ M1

$y' = \lambda e^{3x}(1 + 3x)$, $y'' = \lambda e^{3x}(3 + 3 + 9x) = \lambda e^{3x}(6 + 9x)$ M1

$\lambda e^{3x}[(6 + 9x) - (1 + 3x) - 6x] = 5\lambda e^{3x} = 10e^{3x} \Rightarrow \lambda = 2$ A1

$\boxed{y = 2xe^{3x}}$ A1

(b) CF: $Ae^{3x} + Be^{-2x}$. $\boxed{y = Ae^{3x} + Be^{-2x} + 2xe^{3x}}$ M1 A1

(c) $x = 0$, $y = 0$: $A + B = 0$ B1

$y' = 3Ae^{3x} - 2Be^{-2x} + 2e^{3x} + 6xe^{3x}$ M1

$x = 0$, $y' = 1$: $3A - 2B + 2 = 1 \Rightarrow 3A + 2A = -1 \Rightarrow A = -\frac{1}{5}$, $B = \frac{1}{5}$ M1

$\boxed{y = -\frac{1}{5}e^{3x} + \frac{1}{5}e^{-2x} + 2xe^{3x}}$ A1

(Total for Question 7 is 11 marks)

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8.

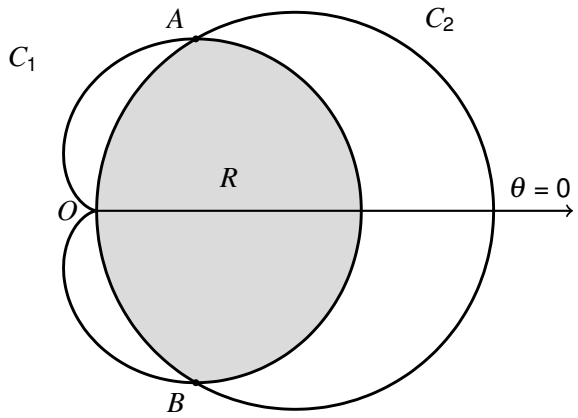

Figure 1

Figure 1 shows the curves C_1 and C_2 with polar equations

$$C_1 : r = 1 + \cos\theta, \quad -\pi < \theta \leqslant \pi \qquad C_2 : r = 3\cos\theta, \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$$

The curves meet at the pole and at the points A and B

(a) Find the polar coordinates of A and the polar coordinates of B

(3)

The region R, shown shaded in Figure 1, lies inside both C_1 and C_2

(b) Use calculus to find the exact area of R

(8)

(a) $1 + \cos\theta = 3\cos\theta \Rightarrow \cos\theta = \frac{1}{2}$, so $\theta = \pm\frac{\pi}{3}$ and $r = 1 + \frac{1}{2} = \frac{3}{2}$ M1 A1

$A\left(\frac{3}{2}, \frac{\pi}{3}\right)$ $B\left(\frac{3}{2}, -\frac{\pi}{3}\right)$ A1

(b) Use symmetry. Inner curve: C_1 for $0 \leqslant \theta \leqslant \frac{\pi}{3}$, C_2 for $\frac{\pi}{3} \leqslant \theta \leqslant \frac{\pi}{2}$ M1

$\frac{1}{2}\int (1 + \cos\theta)^2\, d\theta = \frac{1}{2}\int \left(1 + 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta$ M1

$= \frac{1}{2}\left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi/3}$ A1

$= \frac{1}{2}\left(\frac{\pi}{2} + \sqrt{3} + \frac{\sqrt{3}}{8}\right) = \frac{\pi}{4} + \frac{9\sqrt{3}}{16}$ A1

$\frac{1}{2}\int_{\pi/3}^{\pi/2} 9\cos^2\theta\, d\theta = \frac{9}{4}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\pi/3}^{\pi/2} = \frac{9}{4}\left(\frac{\pi}{2} - \frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$ M1

$= \frac{3\pi}{8} - \frac{9\sqrt{3}}{16}$ A1

Area of $R = 2\left(\frac{\pi}{4} + \frac{3\pi}{8}\right) = 5\pi/4$ (the $\sqrt{3}$ terms cancel) M1 A1

(Total for Question 8 is 11 marks)

TOTAL FOR PAPER IS 75 MARKS

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