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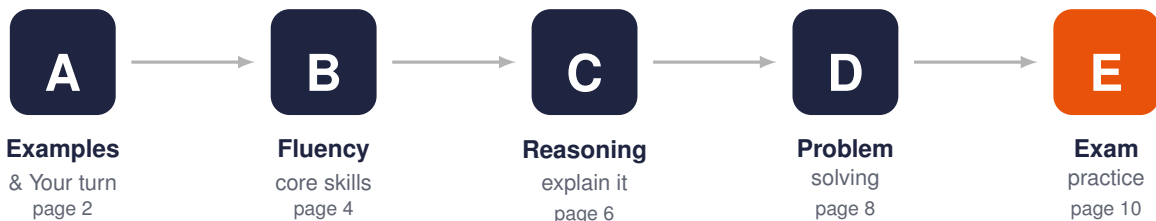
A-LEVEL MATHEMATICS • PURE 4

# Partial Fractions

## Worksheet

Pearson Edexcel International A Level (WMA14) & A Level (9MA0) • Calculator allowed: show every step

Free Partial Fractions worksheet for A-Level and International A-Level Maths, with full worked solutions



Tap a section to jump straight to it

**By the end of this worksheet you can:** split a fraction with distinct linear factors by substituting values • find constants by comparing coefficients • handle a repeated factor with both  $\frac{A}{x-a}$  and  $\frac{B}{(x-a)^2}$  • divide first when a fraction is improper

- *Stretch*: choose the form, find unknown coefficients, and use partial fractions for asymptotes, gradients and maxima.

### Which questions are for me?

**CORE** The methods every A-level student must secure. Start here.

**STRETCH** Improper fractions with repeated factors, forms you choose yourself, unknown coefficients, and partial fractions put to use.

Every section has both, clearly labelled. Questions get harder as you go, and every skill in Sections A to D is needed in Section E.

### Track your progress

| Section       | B  | C  | D  | E  |
|---------------|----|----|----|----|
| Core marks    | 23 | 10 | 13 | 11 |
| Stretch marks | 16 | 14 | 17 | 8  |
| My score      |    |    |    |    |

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## A

## Worked examples &amp; Your turn

Read each example carefully. Cover it up, then answer the question beside it using the same method.

## CORE

Secure the method

## Worked example A1: distinct linear factors

Express  $\frac{5x+4}{(x-1)(x+2)}$  in partial fractions.

Write the identity, then substitute the value that makes each bracket zero.

$$5x+4 \equiv A(x+2)+B(x-1)$$

$$x=1: 9=3A, \text{ so } A=3$$

$$x=-2: -6=-3B, \text{ so } B=2$$

$$\frac{3}{x-1} + \frac{2}{x+2}$$

## Your turn A1

Express  $\frac{x+11}{(2x-3)(x+1)}$  in partial fractions.

## Worked example A2: comparing coefficients

Express  $\frac{8x+1}{(2x+1)(x-1)}$  in partial fractions by comparing coefficients.

$$8x+1 \equiv A(x-1)+B(2x+1) = (A+2B)x + (B-A)$$

Equate the coefficients of  $x$ , then the constants.

$$A+2B=8 \text{ and } B-A=1, \text{ so } 3B=9: B=3, A=2$$

$$\frac{2}{2x+1} + \frac{3}{x-1}$$

## Your turn A2

Express  $\frac{13x-7}{(3x-2)(x+1)}$  in partial fractions by comparing coefficients.

## Worked example A3: repeated factors

Express  $\frac{x^2+8x-2}{(x-2)(x+1)^2}$  in partial fractions.

A squared factor needs two terms:  $\frac{B}{x+1}$  and  $\frac{C}{(x+1)^2}$

$$x^2+8x-2 \equiv A(x+1)^2+B(x-2)(x+1)+C(x-2)$$

$$x=2: 18=9A, A=2; \quad x=-1: -9=-3C, C=3$$

$$x^2: 1=A+B, \text{ so } B=-1$$

$$\frac{2}{x-2} - \frac{1}{x+1} + \frac{3}{(x+1)^2}$$

## Your turn A3

Express  $\frac{2x^2-7x+28}{(2x-1)(x-3)^2}$  in partial fractions.



## STRETCH

**Worked example A4: improper, equal degrees**

Express  $\frac{2x^2+1}{(x-1)(x+2)}$  in partial fractions.

*Equal degrees, so there is a constant term:  $A = 2$*

$$2x^2+1 \equiv 2(x-1)(x+2)+B(x+2)+C(x-1)$$

$$x=1: 3=3B; \quad x=-2: 9=-3C$$

$$2 + \frac{1}{x-1} - \frac{3}{x+2}$$

**Your turn A4**

Express  $\frac{3x^2-3x+2}{(x+1)(x-3)}$  in partial fractions.

**Worked example A5: cubic over quadratic**

Express  $\frac{2x^3-3x^2+x+3}{(x-2)(x+1)}$  in partial fractions.

*Divide by  $x^2-x-2$ : quotient  $2x-1$ , remainder  $4x+1$*

$$4x+1 \equiv C(x+1)+D(x-2)$$

$$x=2: C=3; \quad x=-1: D=1$$

$$2x-1 + \frac{3}{x-2} + \frac{1}{x+1}$$

**Your turn A5**

Express  $\frac{x^3+5x^2+4x-14}{(x+3)(x-1)}$  in partial fractions.

**Worked example A6: choosing the form**

Express  $\frac{x^2+2}{x^3-x^2}$  in partial fractions.

*$x^3-x^2 = x^2(x-1)$ : terms in  $x$ ,  $x^2$  and  $x-1$*

$$x^2+2 \equiv Ax(x-1)+B(x-1)+Cx^2$$

$$x=0: B=-2; \quad x=1: C=3; \quad x^2: A+C=1, A=-2$$

$$-\frac{2}{x} - \frac{2}{x^2} + \frac{3}{x-1}$$

**Your turn A6**

Express  $\frac{x^2+3x+1}{2x^3+x^2}$  in partial fractions.

**Worked example A7: improper, repeated factor**

Express  $\frac{x^3+1}{x(x-1)^2}$  in partial fractions.

*Equal degrees, so a constant 1:  $x^3+1 \equiv x(x-1)^2 + A(x-1)^2 + Bx(x-1) + Cx$*

$$x=0: A=1; \quad x=1: C=2; \quad x^2: -2+A+B=0, B=1$$

$$1 + \frac{1}{x} + \frac{1}{x-1} + \frac{2}{(x-1)^2}$$

**Your turn A7**

Express  $\frac{2x^3-x-5}{(x-2)(x+1)^2}$  in partial fractions.



**B****Fluency**

Each question is a designed sequence. From one part to the next, ask yourself: what changed, and what did it do to the answer?

**CORE***Secure the method*

**B1** Write down the form of the partial fractions for each expression. Do not find the constants.

(a)  $\frac{3x+1}{(x+2)(x-5)}$

(b)  $\frac{x^2+1}{(x-1)(x+3)(2x+1)}$

(c)  $\frac{4x-3}{(x+1)^2(x-2)}$

(d)  $\frac{x^2+7}{(x-1)(x+4)}$

[4 marks]

**B2** Express each fraction in partial fractions.

(a)  $\frac{5x+1}{(x+1)(x-1)}$

(b)  $\frac{x+7}{(x-3)(x+2)}$

(c)  $\frac{14}{(2x+1)(x-3)}$

(d)  $\frac{4x^2-3x-4}{x(x-1)(x+2)}$

[4 marks]

**B3** Use comparing coefficients in each part.

(a) Express  $\frac{5x+7}{(x+1)(x+2)}$  in partial fractions.

(b) Express  $\frac{x-10}{(2x+1)(x-3)}$  in partial fractions.

(c) Find  $a$  and  $b$  such that  $\frac{ax+b}{(x-1)(x+4)} \equiv \frac{2}{x-1} + \frac{3}{x+4}$

(d) Find  $A$ ,  $B$  and  $C$  such that  $3x^2+2x+5 \equiv A(x+1)^2+B(x+1)+C$

[4 marks]

**B4** Express each fraction in partial fractions.

(a)  $\frac{x+5}{(x+1)^2}$

(b)  $\frac{3x^2-3x-9}{(x+2)(x-1)^2}$

(c)  $\frac{2x+1}{x(2x-1)^2}$

(d)  $\frac{x+4}{x^2(x+2)}$

[4 marks]

**B5** Express each fraction as a polynomial plus partial fractions.

(a)  $\frac{x^2+3x}{(x+1)(x+2)}$

(b)  $\frac{2x^2-x+3}{x(x-1)}$

(c)  $\frac{x^3-2}{x(x+1)}$

[3 marks]



**B6** Factorise each denominator, decide the form, and express each fraction in partial fractions.

(a)  $\frac{x+13}{x^2+x-6}$

(b)  $\frac{6x-2}{x^2-1}$

(c)  $\frac{4x+1}{x^3-x}$

(d)  $\frac{2x^2+5}{x^2-2x+1}$

[4 marks]

### STRETCH

**B7** (a) Express  $\frac{x^3}{(x+1)^2}$  as a polynomial plus partial fractions.

(b) Express  $\frac{x^4+1}{x^2(x+1)}$  as a polynomial plus partial fractions.

(c) Express  $\frac{6x^2+x-1}{(2x+1)(x-1)}$  as simply as possible.

(d) Find the value of  $k$  for which  $\frac{x^2+kx+8}{(x-1)(x-2)(x-3)}$  has no term in  $\frac{1}{x-2}$

[4 marks]

**B8** Express each fraction in partial fractions, where  $a$  and  $k$  are non-zero constants.

(a)  $\frac{1}{x^2-a^2}$

(b)  $\frac{1}{x(x+k)}$

(c)  $\frac{y^2+1}{y^3-y}$

(d)  $\frac{4-x}{2x^2+5x-3}$

[4 marks]

**B9** (a) Given that  $\frac{x^2+ax+5}{(x-1)(x+2)^2} \equiv \frac{2}{x-1} - \frac{1}{x+2} + \frac{C}{(x+2)^2}$ , find  $a$  and  $C$

(b) Find  $A$ ,  $B$  and  $C$  such that  $2x^2-5x+4 \equiv A(x-1)^2+B(x-1)+C$

(c) Hence write  $\frac{2x^2-5x+4}{(x-1)^2}$  in partial fractions.

(d) Express  $\frac{1}{(x+1)(x+2)(x+3)}$  in partial fractions, and explain why the numerators add to 0

[4 marks]

**B10**  $f(x) = \frac{2x^2+1}{(x-1)(x+2)}$ ,  $x \neq 1$ ,  $x \neq -2$

(a) Express  $f(x)$  in the form  $A + \frac{B}{x-1} + \frac{C}{x+2}$

(b) Write down the equations of the three asymptotes of  $y = f(x)$

(c) Is  $f(x)$  above or below 2 for large positive  $x$ ? Use (a).

(d) Find the value of  $x$  for which  $f(x) = 2$

[4 marks]



**C****Reasoning**

Show your working and explain your thinking. A correct answer without a reason does not score full marks.

**CORE***Secure the method*

**C1** Asha writes  $\frac{3x+1}{(x-1)(x+1)^2} \equiv \frac{A}{x-1} + \frac{B}{(x+1)^2}$

Explain why this form cannot work for every value of  $x$ , and write down the correct form.

**[2 marks]**

**C2** Ben writes  $\frac{x^2+1}{(x-1)(x+2)} \equiv \frac{A}{x-1} + \frac{B}{x+2}$

Explain what Ben has missed, and write down the correct form.

**[2 marks]**

**C3** For  $\frac{4x+1}{(x-2)(x+1)^2} \equiv \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$ , Cara writes

$$4x+1 \equiv A(x+1) + B(x-2)(x+1) + C(x-2)$$

and substitutes  $x = 2$  to get  $A = 3$ . Find her error, correct the identity, and find the correct value of  $A$

**[3 marks]**

**C4** Dev says: "In  $5x+4 \equiv A(x+2) + B(x-1)$  you cannot substitute  $x = 1$ , because the fraction  $\frac{5x+4}{(x-1)(x+2)}$  is not defined when  $x = 1$ "

Explain why substituting  $x = 1$  is valid, and use it to find  $A$

**[3 marks]**

## STRETCH

- C5** Eve writes  $\frac{x^2+2x}{(x+2)(x-3)} \equiv A + \frac{B}{x+2} + \frac{C}{x-3}$ , and finds  $B = 0$ . She thinks she has made a mistake.

Explain why  $B = 0$  is correct, and write the fraction in its simplest form.

[3 marks]

- C6** Two students write  $\frac{3x+2}{(x-1)^2}$  in different forms:  $\frac{A}{x-1} + \frac{B}{(x-1)^2}$  and  $\frac{Px+Q}{(x-1)^2}$

Find  $A$ ,  $B$ ,  $P$  and  $Q$ , and explain why the first form is the one used for partial fractions.

[3 marks]

- C7** Sam writes  $\frac{x+7}{(x-1)(x+3)} \equiv \frac{3}{x-1} + \frac{2}{x+3}$ . He checks  $x = 0$ , and both sides equal  $-\frac{7}{3}$

Explain why this check does not prove his answer, show that it is wrong, and correct it.

[3 marks]

- C8** (a) Prove that, for constants  $a$  and  $b$  with  $a \neq b$ ,

$$\frac{1}{(x+a)(x+b)} \equiv \frac{1}{b-a} \left( \frac{1}{x+a} - \frac{1}{x+b} \right)$$

(3)

- (b) Hence write  $\frac{1}{x^2+5x+6}$  and  $\frac{1}{4x^2-1}$  in partial fractions.

(2)

[5 marks]



## D

## Problem solving

You decide the method. Read the whole question first, and show every step.

## CORE

Secure the method

- D1** In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.

The function  $f$  is defined by  $f(x) = \frac{x^2 - 7x + 32}{(x+2)(x-3)^2}$ ,  $x > 3$

Given that  $f(x)$  can be written in the form  $\frac{A}{x+2} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$ , where  $A$ ,  $B$  and  $C$  are constants, find the values of  $A$ ,  $B$  and  $C$ . Show clearly how you find each constant, and check your answer by substituting  $x = 0$  into both sides. [3 marks]

- D2** Figure 1 shows a sketch of the curve  $C$  with equation

$$y = \frac{x^2 + 2x + 3}{(x-1)(x+2)}, \quad x \neq 1, x \neq -2$$

The lines with equations  $x = -2$ ,  $x = 1$  and  $y = 1$  are asymptotes to  $C$ . The curve meets the  $y$ -axis at  $\left(0, -\frac{3}{2}\right)$  and crosses the asymptote  $y = 1$  at the point  $P$

(a) Express  $y$  in the form  $A + \frac{B}{x-1} + \frac{C}{x+2}$ , where  $A$ ,  $B$  and  $C$  are integers to be found.

(3)

(b) Hence, using algebra, find the exact coordinates of  $P$

(1)

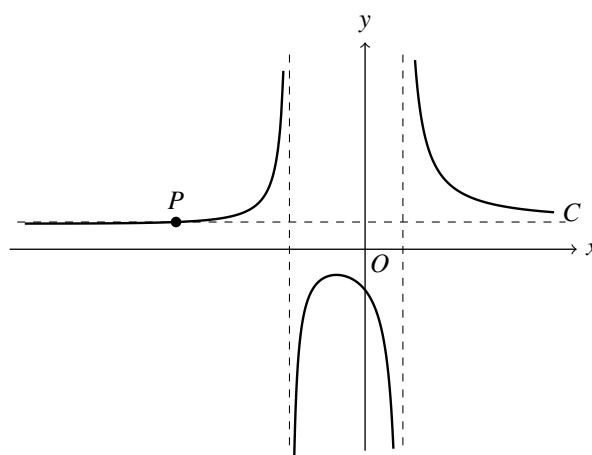


Figure 1

[4 marks]

- D3** In this question  $k$  is a constant.

For one value of  $k$ , the two partial fractions of  $\frac{kx+4}{(x+1)(x-2)}$  have the same numerator, so that

$$\frac{kx+4}{(x+1)(x-2)} \equiv \frac{A}{x+1} + \frac{A}{x-2}$$

where  $A$  is also a constant. Find the value of  $k$  and the value of  $A$ , showing your method clearly, and hence write down the partial fractions of the expression for this value of  $k$  [3 marks]

- D4** The function  $h$  is defined by  $h(x) = \frac{x^2 + 10x + 1}{x^3 - x}$ ,  $x > 1$

By first factorising the denominator completely, express  $h(x)$  in partial fractions. You should write down the form you are using before you find the constants, and show clearly how each constant is found. Check your answer by substituting  $x = 2$  into both sides of your identity. [3 marks]



## STRETCH

- D5** The function  $f$  is defined by  $f(x) = \frac{ax^2 + bx + c}{(x+1)(x-2)}$ ,  $x \neq -1, x \neq 2$ , where  $a, b$  and  $c$  are constants.

The curve with equation  $y = f(x)$  passes through the point  $(0, 1)$ . Given also that

$$f(x) \equiv 3 + \frac{2}{x+1} + \frac{B}{x-2}$$

where  $B$  is a constant, find the value of  $B$  and the values of  $a, b$  and  $c$ . Give each answer as an integer, and show clearly how you use the given point. [4 marks]

- D6** The function  $f$  is defined by  $f(x) = \frac{5x^2 + 6x + 2}{x(x+1)^2}$ ,  $x > 0$

(a) Express  $f(x)$  in partial fractions, giving each constant as an integer. (3)

(b) Hence prove that  $f(x) > \frac{2}{x}$  for all values of  $x$  in the domain of  $f$ . You must make your reasoning clear, and a sketch alone is not a proof. (1)

[4 marks]

- D7** In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.

The curve  $C$  has equation  $y = \frac{x^3 - 3x + 4}{x(x-1)^2}$ ,  $x \neq 0, x \neq 1$

(a) Express  $y$  in the form  $A + \frac{B}{x} + \frac{D}{x-1} + \frac{E}{(x-1)^2}$ , where  $A, B, D$  and  $E$  are integers to be found. (3)

(b) The line with equation  $y = 1$  is an asymptote to  $C$ . Using your answer to part (a), or otherwise, show that  $C$  never meets this asymptote. (1)

[4 marks]

- D8** The function  $f$  is defined by  $f(x) = \frac{3x+7}{x+2}$ ,  $x > -2$

(a) Express  $f(x)$  in the form  $A + \frac{B}{x+2}$ , where  $A$  and  $B$  are integers to be found. (2)

(b) Hence find the range of  $f$ , giving your answer as an inequality. (1)

(c) Find an expression for  $f^{-1}(x)$  in the form  $P + \frac{Q}{x+R}$ , where  $P, Q$  and  $R$  are integers, and state the domain of  $f^{-1}$ . (2)

[5 marks]



**E****Exam practice**

Exam-style questions in the style of WMA14. Write your answers in the spaces, and show every step.

**CORE**

Exam questions for everyone

- E1** In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.

The function  $f$  is defined by  $f(x) = \frac{3x^2 - 6x + 9}{(x+1)(x-2)^2}$ ,  $x > 2$

- (a) Express  $f(x)$  in the form  $\frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$  where  $A$ ,  $B$  and  $C$  are constants to be found. Show clearly how you find each constant.

(3)

- (b) Using your answer to part (a) and differentiation, find the exact value of the gradient of the curve with equation  $y = f(x)$  at the point where  $x = 3$ , giving your answer as a fraction in its simplest form.

(2)

(Total for Question E1 is 5 marks)

- E2** The function  $g$  is defined by

$$g(x) = \frac{4x^2 + 3x + 2}{(2x-1)(x+1)}, \quad x > \frac{1}{2}$$

- (a) Express  $g(x)$  in the form  $A + \frac{B}{2x-1} + \frac{C}{x+1}$  where  $A$ ,  $B$  and  $C$  are constants to be found. Show clearly how you find each constant.

(4)

- (b) Using your answer to part (a), explain why the curve with equation  $y = g(x)$  never meets the line with equation  $y = 2$  for any value of  $x$  in the domain of  $g$ . You must make your reasoning clear.

(2)

(Total for Question E2 is 6 marks)



## STRETCH

- E3** A chemical is added to a tank of water. The concentration of the chemical,  $C$  mg per litre,  $t$  hours after it is added, is modelled by the equation

$$C = \frac{15t}{(t+1)(t+4)}, \quad t \geq 0$$

Figure 2 shows a sketch of the graph of  $C$  against  $t$ . According to the model, the concentration reaches its maximum value at the point  $M$

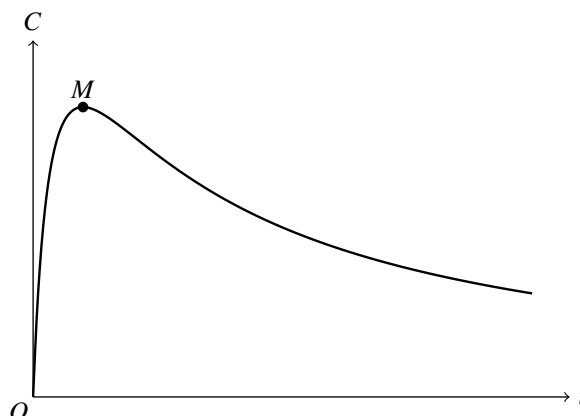


Figure 2

- (a) Express  $\frac{15t}{(t+1)(t+4)}$  in partial fractions.

(3)

- (b) Using your answer to part (a) and differentiation, find the exact time at which the concentration is greatest, and the maximum concentration.

(4)

- (c) State, with a reason, what happens to the concentration in the long term, according to the model.

(1)

(Total for Question E3 is 8 marks)

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