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A-LEVEL MATHEMATICS • FURTHER PURE 3

Further Coordinate Systems

Worksheet

Pearson Edexcel International A Level (WFM03) • Calculator allowed: show every step

Free Further Coordinate Systems worksheet for A-Level and International A-Level Maths, with full worked solutions

A **Examples** & Your turn page 2

B **Fluency** core skills page 4

C **Reasoning** explain it page 6

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Tap a section to jump straight to it

By the end of this worksheet you can: find eccentricities, foci, directrices and asymptotes • use parametric forms • *Stretch:* tangents and normals by calculus, the focus-directrix property, and loci.

Which questions are for me?

CORE The methods every A-level student must secure. Start here.

STRETCH Ellipses and hyperbolas: eccentricity, foci and directrices, tangents and normals by calculus, and loci.

Every section has both, clearly labelled. Questions get harder as you go, and every skill in Sections A to D is needed in Section E.

Track your progress

Section	B	C	D	E
Core marks	23	10	13	13
Stretch marks	16	14	17	9
My score				

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Name ..... Class ..... Date .....

A Worked examples & Your turn

Read each example carefully. Cover it up, then answer the question beside it using the same method.

CORE *Secure the method*

Worked example A1: an ellipse

For $\frac{x^2}{25}+\frac{y^2}{16}=1$ find e, the foci and the directrices.

$16=25\left(1-e^2\right)$, *so* $e=\frac{3}{5}$

$(\pm 3,0),\ x=\pm\frac{25}{3}$

Your turn A1

For $\frac{x^2}{169}+\frac{y^2}{144}=1$ find e, the foci and the directrices.

Worked example A2: a hyperbola

For $\frac{x^2}{16}-\frac{y^2}{9}=1$ find e, the foci, the directrices and the asymptotes.

$9=16\left(e^2-1\right)$, *so* $e=\frac{5}{4}$

$(\pm 5,0),\ x=\pm\frac{16}{5},\ y=\pm\frac{3}{4}x$

Your turn A2

For $\frac{x^2}{9}-\frac{y^2}{16}=1$ find e, the foci, the directrices and the asymptotes.

Worked example A3: parametric forms

Show that $(3\cos\theta,2\sin\theta)$ lies on $\frac{x^2}{9}+\frac{y^2}{4}=1$

$\cos^2\theta+\sin^2\theta=1$

on the ellipse

Your turn A3

Show that $(4\sec\theta,3\tan\theta)$ lies on $\frac{x^2}{16}-\frac{y^2}{9}=1$

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STRETCH

Worked example A4: a tangent

Find the tangent to $\frac{x^2}{25}+\frac{y^2}{4}=1$ at $(5\cos\theta, 2\sin\theta)$

$\frac{dy}{dx}=\frac{2\cos\theta}{-5\sin\theta}$

$5y\sin\theta-10\sin^2\theta=-2x\cos\theta+10\cos^2\theta$

$2x\cos\theta+5y\sin\theta=10$

Your turn A4

Find the tangent to $\frac{x^2}{9}-\frac{y^2}{4}=1$ at $(3\sec\theta, 2\tan\theta)$

Worked example A5: a normal

Find the normal to $\frac{x^2}{25}+\frac{y^2}{4}=1$ at $(5\cos\theta, 2\sin\theta)$

Gradient $\frac{5\sin\theta}{2\cos\theta}$

$5x\sin\theta-2y\cos\theta=21\sin\theta\cos\theta$

Your turn A5

Find the normal to $\frac{x^2}{16}+\frac{y^2}{9}=1$ at $(4\cos\theta, 3\sin\theta)$

Worked example A6: the focus-directrix property

P lies on $\frac{x^2}{25}+\frac{y^2}{16}=1$ with $x=4$. Find PS, where S is $(3,0)$

$PS=e\times PM=\frac{3}{5}\left(\frac{25}{3}-4\right)$

$\frac{13}{5}$

Your turn A6

P lies on $\frac{x^2}{16}-\frac{y^2}{9}=1$ with $x=6$. Find PS, where S is $(5,0)$

Worked example A7: a locus

Find the locus of P where $PS=\frac{1}{2}PM$, $S(2,0)$ and M on $x=8$

$e=\frac{1}{2}$, $ae=2$: $a=4$, $b^2=16\left(1-\frac{1}{4}\right)$

$\frac{x^2}{16}+\frac{y^2}{12}=1$

Your turn A7

Find the locus of P where $PS=2PM$, $S(4,0)$ and M on $x=1$

B Fluency

Each question is a designed sequence. From one part to the next, ask yourself: what changed, and what did it do to the answer?

CORE *Secure the method*

B1 For the ellipse $4x^2 + 9y^2 = 36$, find

(a) a and b

(b) the eccentricity

(c) the foci

(d) the directrices

[4 marks]

B2 For the hyperbola $x^2 - 4y^2 = 4$, find

(a) the eccentricity

(b) the foci

(c) the directrices

(d) the asymptotes

[4 marks]

B3 Find the equation of

(a) the ellipse with foci $(\pm 4, 0)$ and $e = \frac{4}{5}$

(b) the ellipse with directrices $x = \pm 9$ and $e = \frac{1}{3}$

(c) the hyperbola with foci $(\pm 10, 0)$ and $e = \frac{5}{4}$

(d) the hyperbola through $(3, 0)$ with asymptotes $y = \pm 2x$

[4 marks]

B4

(a) Find the tangent to $\frac{x^2}{8} + \frac{y^2}{2} = 1$ at $(2, 1)$

(b) Find the tangent to $\frac{x^2}{9} - \frac{y^2}{3} = 1$ at $(6, 3)$

(c) Find the gradient of the normal in (a)

(d) Find the normal in (a)

[4 marks]

B5 Find a Cartesian equation for

(a) $x = 2\cos\theta$, $y = 5\sin\theta$

(b) $x = 3\cosh t$, $y = 2\sinh t$

(c) $x = 4\sec\theta$, $y = \tan\theta$

[3 marks]

B6 The line $y = x + c$ and the ellipse $\frac{x^2}{4} + y^2 = 1$

(a) Find the quadratic in x for their meeting points.

(b) State the condition for the line to be a tangent.

(c) Find the values of c

(d) Find the point of contact for the positive c

[4 marks]

STRETCH

B7 $P(3\cosh t, 2\sinh t)$ lies on $\frac{x^2}{9} - \frac{y^2}{4} = 1$

(a) Find $\frac{dy}{dx}$ at P

(b) Find the tangent at P

(c) Find where it meets the x-axis.

(d) Find where it meets the y-axis.

[4 marks]

B8 $P(x, y)$ lies on $\frac{x^2}{49} + \frac{y^2}{24} = 1$

(a) Find the foci S and S'

(b) Find PS in terms of x

(c) Find PS' in terms of x

(d) Show that $PS + PS'$ is constant.

[4 marks]

B9 $P(3\cos\theta, \sin\theta)$ lies on $\frac{x^2}{9} + y^2 = 1$

(a) Find the normal at P

(b) Find where it meets the x-axis.

(c) Find where it meets the y-axis.

(d) Find both intercepts when $\theta = \frac{\pi}{6}$

[4 marks]

B10 $P(2\cos\theta, 3\sin\theta)$ lies on $\frac{x^2}{4} + \frac{y^2}{9} = 1$

(a) Find the midpoint of OP

(b) Find its locus.

(c) Find the midpoint of P and $A(2, 0)$

(d) Find and describe its locus.

[4 marks]

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C Reasoning

Show your working and explain your thinking. A correct answer without a reason does not score full marks.

CORE *Secure the method*

C1 For $\frac{x^2}{25}+\frac{y^2}{9}=1$, Amir writes $e^2=1+\frac{9}{25}$. Explain his error, and find e **[2 marks]**

C2 Bea finds $e^2=\frac{5}{9}$ for an ellipse and rejects $e=\frac{\sqrt{5}}{3}$ because "it is bigger than 1". Explain her error.

[2 marks]

C3 The tangent at P has gradient $-\frac{4\cos\theta}{5\sin\theta}$. Cal writes the perpendicular line through O as $y=\frac{5\sin\theta}{4\cos\theta}$

Explain his error, and correct it. **[3 marks]**

C4 For $\frac{x^2}{36}+\frac{y^2}{27}=1$, Dev gives the directrices as the points $(\pm 12, 0)$

Explain his error, and give the directrices correctly. **[3 marks]**

STRETCH

C5 At $(4\sec\theta, 3\tan\theta)$ on $\frac{x^2}{16}-\frac{y^2}{9}=1$, Eve writes $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3\sec^2\theta}{4\sec\theta}$

Explain her error, and find $\frac{\mathrm{d}y}{\mathrm{d}x}$ **[3 marks]**

C6 For $\frac{x^2}{9}-\frac{y^2}{16}=1$, Fay writes $16=9\left(1-e^2\right)$ and finds no real e

Explain her error, and find e **[3 marks]**

C7 Gus says that the midpoint of OP, for P on $\frac{x^2}{9}+\frac{y^2}{4}=1$, moves on a circle.

Show that he is wrong, and find the locus. **[3 marks]**

C8 **(a)** Use implicit differentiation to show that the tangent to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at (x_1, y_1) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

(3)

(b) Show that the line $y=mx+c$ is a tangent to this ellipse when $c^2=a^2m^2+b^2$

(2)

[5 marks]

D Problem solving

You decide the method. Read the whole question first, and show every step.

CORE *Secure the method*

D1 The hyperbola H has equation $kx^2 - y^2 = 16$, where k is a positive constant. The foci of H lie on the x-axis, symmetrically placed about the origin O

Given that the point $(5, 0)$ is a focus of H,

(a) determine the value of k, showing your working clearly and stating any formula that you use

(2)

(b) hence determine an equation of the directrix of H that corresponds to this focus.

(1)

[3 marks]

D2 The ellipse E has its centre at the origin and its major axis along the x-axis. The foci of E have coordinates $\left(\pm 2\sqrt{3}, 0\right)$ and the directrices of E have equations $x = \pm\dfrac{8}{\sqrt{3}}$

Determine, showing your working clearly,

(a) the eccentricity of E, explaining why your answer is consistent with E being an ellipse

(2)

(b) an equation for E, giving your answer in the form $px^2 + qy^2 = r$, where p, q and r are integers.

(2)

[4 marks]

D3 The ellipse E has equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{5} = 1$, where a is a constant with $a > \sqrt{5}$

Given that the line with equation $x = \dfrac{9}{2}$ is a directrix of E, use the relationships between a, b and the eccentricity e to form an equation in a^2. Hence determine the possible values of a^2, and the corresponding values of the eccentricity of E, checking that each value of e is valid for an ellipse. **[3 marks]**

D4 The point $P(6\cos\theta, 4\sin\theta)$, $0 < \theta < \dfrac{\pi}{2}$, lies on the ellipse $\dfrac{x^2}{36} + \dfrac{y^2}{16} = 1$

(a) Using calculus, show that an equation of the tangent to the ellipse at P is $2x\cos\theta + 3y\sin\theta = 12$

(2)

(b) The tangent meets the x-axis at the point A and the y-axis at the point B. Write down the coordinates of A and of B, giving your answers in terms of θ

(1)

[3 marks]

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STRETCH

D5 The point $P(4\cos\theta, 2\sin\theta)$, $0 < \theta < \frac{\pi}{2}$, lies on the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$

(a) Using calculus, show that an equation of the normal to the ellipse at P is $2x\sin\theta - y\cos\theta = 6\sin\theta\cos\theta$

(2)

(b) The normal meets the x-axis at the point N. Determine the exact greatest area of the triangle OPN as θ varies, where O is the origin, and state the value of θ at which it occurs. You must justify your answer.

(2)

[4 marks]

D6 The ellipse E has equation $\frac{x^2}{100} + \frac{y^2}{64} = 1$

(a) Determine the eccentricity of E, the coordinates of the focus S on the positive x-axis, and an equation of the corresponding directrix l

(2)

(b) The point P on E has x-coordinate 5 and a positive y-coordinate, and M is the foot of the perpendicular from P to the directrix l. Determine the exact lengths PS and PM, and verify that $PS = e \times PM$

(2)

[4 marks]

D7 The line with equation $y = x + c$, where c is a constant, meets the ellipse $\frac{x^2}{4} + y^2 = 1$ at two distinct points P and Q

(a) Show that the x-coordinates of P and Q satisfy $5x^2 + 8cx + 4c^2 - 4 = 0$

(1)

(b) Determine, in terms of c, the coordinates of the midpoint M of PQ

(2)

(c) Show that, as c varies, the point M always lies on a straight line through the origin, and state the equation of this line.

(1)

[4 marks]

D8 The hyperbola H has equation $\frac{x^2}{9} - \frac{y^2}{16} = 1$, and S is the focus of H on the positive x-axis. The point $P(3\sec\theta, 4\tan\theta)$, $0 < \theta < \frac{\pi}{2}$, lies on H

(a) Using calculus, show that an equation of the normal to H at P is $3x\sin\theta + 4y = 25\tan\theta$

(3)

(b) The normal meets the x-axis at the point N. By finding SN and SP in terms of θ, show that $SN = e \times SP$, where e is the eccentricity of H

(2)

[5 marks]

E Exam practice

Exam-style questions in the style of WFM03. Write your answers in the spaces, and show every step.

CORE *Exam questions for everyone*

E1 The hyperbola H has equation $4x^2 - 9y^2 = 36$, and O is the origin.

(a) Determine the eccentricity of H

(2)

(b) Hence determine the coordinates of the foci of H and the equations of the directrices of H

(2)

The line l is parallel to one of the asymptotes of H and has equation $y = \frac{2}{3}x + 1$, where the asymptotes are the lines that H approaches for large $|x|$

(c) Show that l meets H at exactly one point, and determine the coordinates of this point.

(2)

(Total for Question E1 is 6 marks)

E2 The ellipse E has equation $\frac{x^2}{25} + \frac{y^2}{9} = 1$

The point $P(5\cos\theta, 3\sin\theta)$, where $0 < \theta < \frac{\pi}{2}$, lies on E

(a) Using calculus, show that an equation of the tangent to E at P is $3x\cos\theta + 5y\sin\theta = 15$

(3)

The tangent to E at P meets the y-axis at the point A, and the normal to E at P, which has equation $5x\sin\theta - 3y\cos\theta = 16\sin\theta\cos\theta$, meets the y-axis at the point B

(b) Write down the coordinates of A and of B

(2)

(c) Determine the least possible length of AB as θ varies, and the value of $\sin\theta$ at which it occurs. You must justify that your value is the least.

(2)

(Total for Question E2 is 7 marks)

STRETCH

E3 The ellipse E has equation $\frac{x^2}{16} + \frac{y^2}{9} = 1$, and O is the origin.

The point $P(4\cos\theta, 3\sin\theta)$, where $0 < \theta < \frac{\pi}{2}$, lies on E

(a) Using calculus, show that an equation of the tangent to E at P is $3x\cos\theta + 4y\sin\theta = 12$

(3)

The tangent meets the x-axis at the point T and the y-axis at the point U

(b) Determine the coordinates of the midpoint of TU in terms of θ

(2)

(c) Show that, as θ varies, the midpoint of TU lies on the curve with equation

$$px^2 + qy^2 = rx^2y^2$$

where p, q and r are integers to be determined.

(4)

(Total for Question E3 is 9 marks)

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