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INTERNATIONAL A LEVEL MATHEMATICS • FURTHER PURE 1 (WFM01)

Further Pure 1 Practice Paper 1

Worked Solutions

AIMING FOR B

Written in the style of the Pearson Edexcel International Advanced Level paper (WFM01/01) • 75 marks

Every question solved in full on the paper itself, the way your teacher would write it out

How to read these solutions. Each page is the real question page, with a full answer written on the answer lines.

Red pen

M1 A1

boxed

the working a full-marks answer shows, with a short note on why each step is taken

the mark earned on that line: M = method, A = accuracy, B = independent mark

the final answer to each part

Sit the paper first. These solutions are most useful after you have tried every question yourself under timed conditions. Mark your paper with the mark scheme, then use this booklet to see exactly how each lost mark could have been earned.

Q	Topic	Marks	Page
1	A matrix that is never singular	6	2
2	Numerical solution of an equation	7	3
3	Roots of a quadratic equation	7	4
4	A complex equation with a conjugate	8	5
5	Series and sigma notation	8	6
6	Transformations using matrices	9	7
7	The normal to a rectangular hyperbola	9	8
8	Complex roots of a quartic	10	9
9	Proof by induction	11	10

Question paper & mark scheme | Exam questions by topic | More practice papers

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1.

$$\mathbf{A} = \begin{pmatrix} k & 3 \\ k-2 & k+1 \end{pmatrix}$$

where k is a real constant.

(a) Find $\det \mathbf{A}$ in terms of k, giving your answer in simplest form.

(1)

(b) Show that $\mathbf{A}$ is non-singular for all real values of k

(2)

(c) Find $\mathbf{A}^{-1}$ in terms of k

(3)

(a) $\det A = k(k+1) - 3(k-2) = k^2 + k - 3k + 6 = \boxed{k^2 - 2k + 6}$ B1

(b) Complete the square: $k^2 - 2k + 6 = (k-1)^2 + 5$ M1

$(k-1)^2 \geqslant 0$, so $\det A \geqslant 5 > 0$: $\det A \neq 0$, so A is non-singular for all k A1

(c) Swap the leading diagonal, change the signs of the other two, divide by det: M1

$A^{-1} = \frac{1}{k^2 - 2k + 6}\begin{pmatrix} k+1 & -3 \\ 2-k & k \end{pmatrix}$ A1 A1

(Total for Question 1 is 6 marks)

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2.

$$f(x) = x^3 - \frac{2}{x} - 6, \qquad x > 0$$

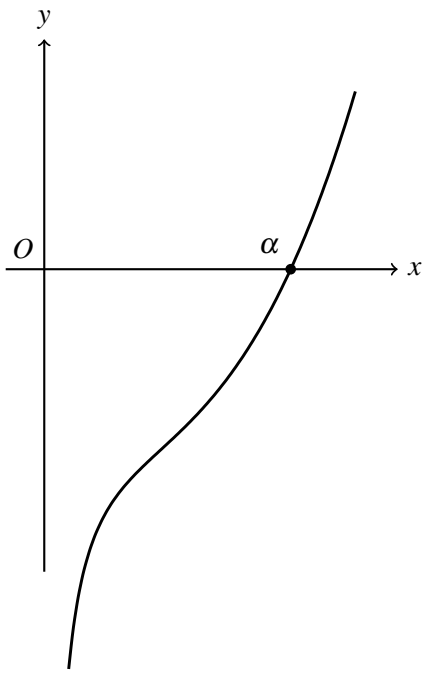

Figure 1

Figure 1 shows part of the curve with equation $y = f(x)$. The equation $f(x) = 0$ has a single root α

(a) Show that α lies in the interval [1.8, 2]

(2)

(b) Taking 2 as a first approximation to α, apply the Newton–Raphson process once to $f(x)$ to find a second approximation to α. Show your working.

(3)

(c) Use linear interpolation once on the interval [1.8, 2] to find another approximation to α, giving your answer to 3 decimal places.

(2)

(a) $f(1.8) = 1.8^3 - 2 \div 1.8 - 6 = -1.279\ldots$ $f(2) = 8 - 1 - 6 = 1$ M1

Sign change, and f is continuous on [1.8, 2] ($x \neq 0$ there), so α is in [1.8, 2] A1

(b) $f'(x) = 3x^2 + 2x^{-2}$ B1

$x_1 = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{1}{12 + 0.5} = 2 - 0.08 = \boxed{1.92}$ M1 A1

(c) $\frac{\alpha - 1.8}{1.279\ldots} = \frac{2 - \alpha}{1}$ (use the size of f(1.8), not its sign) M1

$\alpha = 1.8 + 0.2 \times \frac{1.279\ldots}{2.279\ldots} = 1.91224\ldots$ $\boxed{1.912}$ A1

(Total for Question 2 is 7 marks)

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3. The roots of the equation

$$2x^2 - 3x + 5 = 0$$

are α and β

(a) Write down the value of $\alpha + \beta$ and the value of $\alpha\beta$

(1)

(b) Find the value of $\alpha^2 + \beta^2$

(2)

(c) Find a quadratic equation, with integer coefficients, whose roots are

$$\alpha + \frac{1}{\beta} \quad \text{and} \quad \beta + \frac{1}{\alpha}$$

(4)

(a) $\alpha + \beta = \frac{3}{2}$ $(-b \div a)$ $\alpha\beta = \frac{5}{2}$ $(c \div a)$ B1

(b) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \frac{9}{4} - 5 = \boxed{-\frac{11}{4}}$ M1 A1

(c) Sum $= \alpha + \beta + \frac{1}{\beta} + \frac{1}{\alpha} = \alpha + \beta + \frac{\alpha + \beta}{\alpha\beta}$ M1

$= \frac{3}{2} + \frac{3}{2} \div \frac{5}{2} = \frac{3}{2} + \frac{3}{5} = \frac{21}{10}$ A1

Product $= \alpha\beta + 1 + 1 + \frac{1}{\alpha\beta} = \frac{5}{2} + 2 + \frac{2}{5} = \frac{49}{10}$ M1

$x^2 - \frac{21}{10}x + \frac{49}{10} = 0$, $\times 10$: $\boxed{10x^2 - 21x + 49 = 0}$ A1

(Total for Question 3 is 7 marks)

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4. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

The complex number z satisfies

$$z + 2iz^* = 8 + 7i$$

where z^* is the complex conjugate of z

(a) Find z, giving your answer in the form $a + bi$, where a and b are real.

(4)

(b) Find $|z|$ and $\arg z$, giving the argument in radians to 3 significant figures.

(2)

(c) Find z^2, and show z and z^2 on a single Argand diagram.

(2)

(a) Let $z = a + bi$, so $z^* = a - bi$: $a + bi + 2i(a - bi) = 8 + 7i$ M1

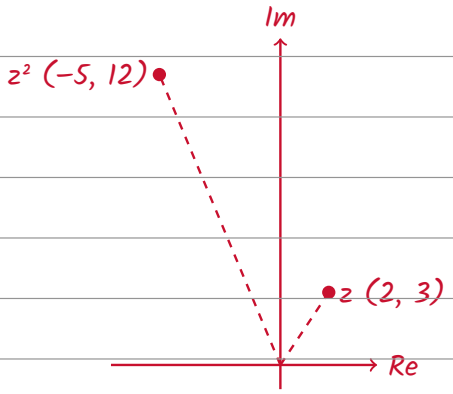
$a + bi + 2ai + 2b = 8 + 7i$ (since $i^2 = -1$)

Real: $a + 2b = 8$ Imaginary: $2a + b = 7$ M1 A1

$2(8 - 2b) + b = 7 \Rightarrow 16 - 3b = 7 \Rightarrow b = 3$, $a = 2$: $\boxed{z = 2 + 3i}$ A1

(b) $|z| = \sqrt{2^2 + 3^2} = \boxed{\sqrt{13}}$ $\arg z = \arctan \frac{3}{2} = \boxed{0.983}$ B1 B1

(c) $z^2 = (2 + 3i)^2 = 4 + 12i + 9i^2 = \boxed{-5 + 12i}$ B1



B1

(Total for Question 4 is 8 marks)

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5. **(a)** Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that, for all positive integers n,

$$\sum_{r=1}^{n} r(r+3) = \tfrac{1}{3}n(n+1)(n+5)$$

(4)

(b) Hence show that

$$\sum_{r=n+1}^{2n} r(r+3) = \tfrac{1}{3}n\left(an^2+bn+c\right)$$

where a, b and c are integers to be found.

(4)

(a) $\sum r(r+3) = \sum r^2 + 3\sum r = \frac{1}{6}n(n+1)(2n+1) + \frac{3}{2}n(n+1)$ M1

$= \frac{1}{6}n(n+1)[(2n+1)+9]$ (take out the common factor) M1

$= \frac{1}{6}n(n+1)(2n+10)$ A1

$= \frac{1}{3}n(n+1)(n+5)$ as required A1

(b) Sum to 2n minus sum to n: $\frac{1}{3}(2n)(2n+1)(2n+5) - \frac{1}{3}n(n+1)(n+5)$ M1

$= \frac{1}{3}n[2(4n^2+12n+5) - (n^2+6n+5)]$ M1

$= \frac{1}{3}n(8n^2+24n+10-n^2-6n-5) = \frac{1}{3}n(7n^2+18n+5)$ A1

$a = 7,\ b = 18,\ c = 5$ A1

(Total for Question 5 is 8 marks)

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6.

$$\mathbf{P} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(a) Describe fully the single geometrical transformation represented by the matrix $\mathbf{P}$

(2)

The transformation Q is an enlargement, scale factor 3, centre $(0, 0)$

(b) Write down the matrix $\mathbf{Q}$ that represents Q

(1)

The transformation represented by $\mathbf{P}$ is followed by the transformation Q, and the combined transformation is represented by the matrix $\mathbf{R}$

(c) Find $\mathbf{R}$

(2)

(d) A triangle T has area 5 cm^2. Find the area of the image of T under the transformation represented by $\mathbf{R}$

(2)

(e) Find the coordinates of the point that is mapped by $\mathbf{R}$ onto the point $(-6, 12)$

(2)

(a) Rotation through 90° anticlockwise about (0, 0) — B1 B1

(b) $Q = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ — B1

(c) P first, then Q, so $R = QP$ (the first transformation goes on the right) — M1

$R = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix}$ — A1

(d) $\det R = 0 \times 0 - (-3)(3) = 9$, so the area is $5 \times 9 =$ 45 cm² — M1 A1

(e) $\begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -6 \\ 12 \end{pmatrix}$: $-3y = -6$, $3x = 12$ — M1

$x = 4$, $y = 2$: (4, 2) — A1

(Total for Question 6 is 9 marks)

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7.

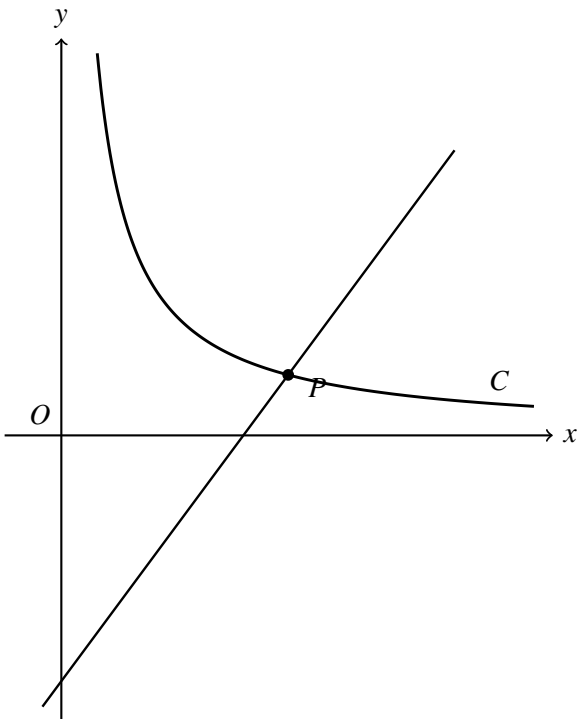

Figure 2

The rectangular hyperbola C has equation $xy = 16$. The point $P\left(4t, \frac{4}{t}\right)$, $t \neq 0$, lies on C, as shown in Figure 2.

(a) Show that an equation of the normal to C at P is

$$t^3x - ty = 4t^4 - 4$$

(4)

The normal to C at the point where $t = 2$ meets the x-axis at A and the y-axis at B

(b) Find the coordinates of A and the coordinates of B

(3)

(c) Find the area of the triangle OAB, where O is the origin.

(2)

(a) $y = 16x^{-1}$, $\frac{dy}{dx} = -16x^{-2} = -\frac{16}{16t^2} = -\frac{1}{t^2}$ at P, so the normal gradient is t^2 M1 A1

$y - \frac{4}{t} = t^2(x - 4t)$; × t: $ty - 4 = t^3x - 4t^4$, so $t^3x - ty = 4t^4 - 4$ M1 A1

(b) $t = 2$: $8x - 2y = 60$. $y = 0$: A(7.5, 0) $x = 0$: B(0, −30) M1 A1 A1

(c) Area $= \frac{1}{2} \times 7.5 \times 30 =$ 112.5 (lengths: 30, not −30) M1 A1

(Total for Question 7 is 9 marks)

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8. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

$$f(z) = z^4 - 4z^3 + 14z^2 - 4z + 13$$

Given that $2 - 3i$ is a root of the equation $f(z) = 0$

(a) write down another complex root of the equation.

(1)

(b) Hence find a quadratic factor of $f(z)$ with real coefficients.

(2)

(c) Find the other two roots of $f(z) = 0$

(5)

(d) Show all four roots of $f(z) = 0$ on a single Argand diagram.

(2)

(a) Real coefficients, so the conjugate is also a root: $\boxed{2 + 3i}$ B1

(b) $(z - 2 + 3i)(z - 2 - 3i) = (z - 2)^2 - (3i)^2 = z^2 - 4z + 4 + 9$ M1

$\boxed{z^2 - 4z + 13}$ A1

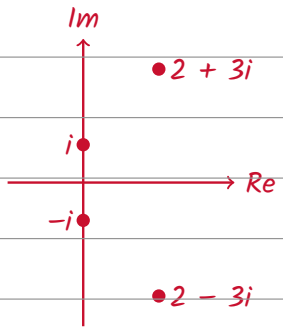
(c) $z^4 - 4z^3 + 14z^2 - 4z + 13 = (z^2 - 4z + 13)(z^2 + az + b)$ M1

z^3: $a - 4 = -4$, so $a = 0$ constant: $13b = 13$, so $b = 1$ A1

Check z^2: $b - 4a + 13 = 14$ ✓ second factor $z^2 + 1$ A1

$z^2 + 1 = 0 \Rightarrow z^2 = -1$ M1

$\boxed{z = i}$ and $\boxed{z = -i}$ A1



B1 B1

(Total for Question 8 is 10 marks)

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9. **(i)** Prove by induction that, for all positive integers n,

$$\sum_{r=1}^{n}(2r-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$$

(5)

(ii) Prove by induction that, for all positive integers n, $3^{2n}+7$ is divisible by 8

(6)

(i) $n = 1$: LHS $= 1^2 = 1$, RHS $= \frac{1}{3}(1)(1)(3) = 1$, so true for $n = 1$ B1

Assume true for $n = k$: $\sum_{r=1}^{k+1}(2r-1)^2 = \frac{1}{3}k(2k-1)(2k+1) + (2k+1)^2$ M1

$= \frac{1}{3}(2k+1)[k(2k-1) + 3(2k+1)] = \frac{1}{3}(2k+1)(2k^2+5k+3)$ M1

$= \frac{1}{3}(k+1)(2k+1)(2k+3) = \frac{1}{3}(k+1)(2(k+1)-1)(2(k+1)+1)$ A1

True for $n = 1$, and if true for $n = k$ then true for $n = k+1$, so true for all $n \geqslant 1$. A1

(ii) $f(n) = 3^{2n} + 7$. $n = 1$: $f(1) = 9 + 7 = 16 = 2 \times 8$, so true for $n = 1$ B1

Assume 8 divides $f(k)$: $f(k+1) - f(k) = 3^{2k+2} - 3^{2k} = 9 \times 3^{2k} - 3^{2k}$ M1

$= 8 \times 3^{2k}$ A1

So $f(k+1) = f(k) + 8 \times 3^{2k}$ M1

$f(k)$ and 8×3^{2k} are both divisible by 8, so $f(k+1)$ is divisible by 8 A1

True for $n = 1$, and if true for $n = k$ then true for $n = k+1$, so true for all $n \geqslant 1$. A1

(Total for Question 9 is 11 marks)

TOTAL FOR PAPER IS 75 MARKS

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