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A-LEVEL MATHEMATICS • FURTHER PURE 2

First-Order Differential Equations

Worksheet

Pearson Edexcel International A Level (WFM02) • Calculator allowed: show every step

Free First-Order Differential Equations worksheet for A-Level and International A-Level Maths, with full worked solutions

A **Examples** & Your turn page 2

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Tap a section to jump straight to it

By the end of this worksheet you can: find integrating factors • find general and particular solutions • describe families of solution curves • *Stretch:* reduce equations by a given substitution, and model with differential equations.

Which questions are for me?

CORE The methods every A-level student must secure. Start here.

STRETCH Integrating factors with trigonometric and power terms, substitutions shown in full, and models with an equilibrium.

Every section has both, clearly labelled. Questions get harder as you go, and every skill in Sections A to D is needed in Section E.

Track your progress

Section	B	C	D	E
Core marks	23	10	13	13
Stretch marks	16	14	17	9
My score				

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A Worked examples & Your turn

Read each example carefully. Cover it up, then answer the question beside it using the same method.

CORE *Secure the method*

Worked example A1: the integrating factor

Solve $\frac{dy}{dx} + 3y = e^{2x}$

IF $= e^{3x}$: $\frac{d}{dx}\left(ye^{3x}\right) = e^{5x}$

$ye^{3x} = \frac{1}{5}e^{5x} + c$

$$y = \tfrac{1}{5}e^{2x} + ce^{-3x}$$

Your turn A1

Solve $\frac{dy}{dx} - y = e^{3x}$

Worked example A2: standard form first

Solve $x\frac{dy}{dx} - y = x^3,\ x > 0$

Divide by x: $\frac{dy}{dx} - \frac{y}{x} = x^2$; *IF* $= e^{-\ln x} = \frac{1}{x}$

$\frac{d}{dx}\left(\frac{y}{x}\right) = x$

$$y = \tfrac{1}{2}x^3 + cx$$

Your turn A2

Solve $x\frac{dy}{dx} + 3y = x^2,\ x > 0$

Worked example A3: a particular solution

Solve $\frac{dy}{dx} + 2y = 6$, given $y = 1$ when $x = 0$

$y = 3 + ce^{-2x}$; $1 = 3 + c$

$$y = 3 - 2e^{-2x}$$

Your turn A3

Solve $\frac{dy}{dx} + y = 2e^x$, given $y = 3$ when $x = 0$

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STRETCH

Worked example A4: a trigonometric IF

Solve $\frac{dy}{dx} + y\cot x = 1,\ 0 < x < \pi$

IF $= e^{\ln \sin x} = \sin x$

$y\sin x = -\cos x + c$

$y = \frac{c - \cos x}{\sin x}$

Your turn A4

Solve $\frac{dy}{dx} + y\tan x = \sec x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$

Worked example A5: a substitution

Show that $z = y^2$ turns $2y\frac{dy}{dx} + y^2 = x$ into $\frac{dz}{dx} + z = x$, and solve it.

$\frac{dz}{dx} = 2y\frac{dy}{dx}$

$z = x - 1 + ce^{-x}$

$y^2 = x - 1 + ce^{-x}$

Your turn A5

Show that $z = y^3$ turns $3y^2\frac{dy}{dx} - y^3 = e^{2x}$ into $\frac{dz}{dx} - z = e^{2x}$, and solve it.

Worked example A6: a family of curves

$x\frac{dy}{dx} + y = 3x^2$ has general solution $y = x^2 + \frac{c}{x}$. Find the curve through $(1, 3)$

$3 = 1 + c$

$y = x^2 + \frac{2}{x}$

Your turn A6

Solve $x\frac{dy}{dx} - 2y = x^3$, and find the curve through $(1, 4)$

Worked example A7: modelling

A drink cools by $\frac{dT}{dt} = -k(T - 20)$, with $T = 90$ at $t = 0$. Find T

Equilibrium 20: $T = 20 + Ae^{-kt}$

$T = 20 + 70e^{-kt}$

Your turn A7

$\frac{dP}{dt} + \frac{P}{10} = 50$ with $P = 0$ at $t = 0$. Find P

B Fluency

Each question is a designed sequence. From one part to the next, ask yourself: what changed, and what did it do to the answer?

CORE *Secure the method*

B1 Find, in simplest form, an integrating factor for

(a) $\frac{dy}{dx} + 4y = x$

(b) $\frac{dy}{dx} + 2xy = x$

(c) $\frac{dy}{dx} + \frac{y}{x} = 1$

(d) $\frac{dy}{dx} - y\tan x = 1$

[4 marks]

B2 Find the general solution of

(a) $\frac{dy}{dx} + 4y = 8$

(b) $\frac{dy}{dx} + 2xy = x$

(c) $\frac{dy}{dx} + \frac{y}{x} = 1$

(d) $\frac{dy}{dx} - y\tan x = 1$

[4 marks]

B3 $\frac{dy}{dx} + 3y = 6x$

(a) Find the general solution.

(b) Find the solution with $y = 0$ at $x = 0$

(c) Find the solution with $y = -\frac{2}{3}$ at $x = 0$, and describe it.

(d) Describe the behaviour of every solution for large x

[4 marks]

B4 Write the left side as a single derivative, and solve

(a) $(x+1)\frac{dy}{dx} + y = x$

(b) $x^2\frac{dy}{dx} + 2xy = 1$

(c) $\cos x\frac{dy}{dx} + y\sin x = \cos^2 x$

(d) $e^x\frac{dy}{dx} + e^x y = 2x$

[4 marks]

B5 $y\frac{dy}{dx} - y^2 = x$ and $z = y^2$

(a) Write $\frac{dz}{dx}$ in terms of y and $\frac{dy}{dx}$

(b) Show that $\frac{dz}{dx} - 2z = 2x$

(c) Solve this equation, and hence find y^2 in terms of x **[3 marks]**

B6 $x\frac{dy}{dx} + y = 2x,\ x > 0$

(a) Find the general solution.

(b) Find the curve through $(1,1)$

(c) Find the curve through $(1,3)$

(d) State the asymptotes of the curves with $c \neq 0$

[4 marks]

STRETCH

B7 $\frac{dy}{dx} - y = xe^x$

(a) Find the integrating factor.

(b) Find the general solution.

(c) Find the solution with $y = 1$ at $x = 0$

(d) Find y at $x = 2$ for that solution.

[4 marks]

B8 $\frac{dy}{dx} = \frac{1}{x+y} - 1$ and $u = x + y$

(a) Show that $\frac{du}{dx} = \frac{1}{u}$

(b) Solve for u

(c) Find c given $y = 2$ at $x = 0$

(d) Write y in terms of x

[4 marks]

B9 $\frac{dy}{dx} - \frac{2y}{x} = -xy^2$ and $z = \frac{1}{y}$

(a) Show that $\frac{dz}{dx} + \frac{2z}{x} = x$

(b) Solve for z

(c) Hence write y in terms of x

(d) Find the solution through $(1,1)$

[4 marks]

B10 Salt in a tank satisfies $\frac{dS}{dt} + \frac{S}{50} = 2$, with $S = 0$ at $t = 0$

(a) Find the integrating factor.

(b) Find S in terms of t

(c) State the long-term amount of salt.

(d) Find the exact time when $S = 50$

[4 marks]

C Reasoning

Show your working and explain your thinking. A correct answer without a reason does not score full marks.

CORE *Secure the method*

C1 For $\frac{dy}{dx} + 2y = x$, Amir writes $e^{2x}\frac{dy}{dx} + 2e^{2x}y = x$. Explain his error, and write the correct equation. **[2 marks]**

C2 From $ye^{2x} = x + c$, Bea writes $y = xe^{-2x} + c$. Explain her error, and correct it. **[2 marks]**

C3 For $x\frac{dy}{dx} + 3y = x$, Cal uses the integrating factor $e^{\int 3\,dx} = e^{3x}$

Explain his error, and solve the equation for $x > 0$ **[3 marks]**

C4 For $\frac{dy}{dx} - y\tan x = 2$, Dev uses the integrating factor $\sec x$

Explain his error, and find the correct integrating factor. **[3 marks]**

STRETCH

C5 After the substitution $z = \frac{1}{y}$, Eve obtains $z = x^2 + c$ and writes $y = \frac{1}{x^2} + \frac{1}{c}$

Explain her error, and write y correctly. **[3 marks]**

C6 For $\frac{dy}{dx} + \frac{2y}{x} = x$, Fay writes the integrating factor as $e^{2\ln x} = 2x$

Explain her error, and solve the equation for $x > 0$ **[3 marks]**

C7 For $\frac{dT}{dt} = -k(T - 20)$, Gus writes $T = Ae^{-kt}$

Explain why this cannot be correct, and give the general solution. **[3 marks]**

C8 **(a)** Show that multiplying $\frac{dy}{dx} + P(x)y = Q(x)$ by $I(x) = e^{\int P(x)\,dx}$ makes the left side $\frac{d}{dx}(I(x)y)$

(3)

(b) Hence show that, for constants $a > 0$ and b, every solution of $\frac{dy}{dx} + ay = b$ tends to $\frac{b}{a}$ as $x \to \infty$

(2)

[5 marks]

D Problem solving

You decide the method. Read the whole question first, and show every step.

CORE *Secure the method*

D1 A curve satisfies the differential equation

$$\frac{dy}{dx} + 2y\tan x = \sin x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

(a) Show that an integrating factor for this differential equation is $\sec^2 x$

(1)

(b) Given that $y = 3$ when $x = 0$, determine the particular solution of the differential equation, giving your answer in the form $y = f(x)$, where $f(x)$ is in terms of $\cos x$ only.

(2)

[3 marks]

D2 By first finding an integrating factor, determine the particular solution of the differential equation

$$\frac{dy}{dx} + \frac{3y}{x} = \frac{e^x}{x^2}, \quad x > 0$$

for which $y = 2$ when $x = 1$, giving your answer in the form $y = f(x)$ **[4 marks]**

D3 Showing all stages of your working, and making the use of the chain rule clear, show that the substitution $z = y^3$ transforms the differential equation

$$y^2\frac{dy}{dx} + \frac{y^3}{x} = x, \quad x > 0 \quad \text{(I)}$$

into the differential equation

$$\frac{dz}{dx} + \frac{3z}{x} = 3x \quad \text{(II)}$$

[3 marks]

D4 A curve C has an equation $y = f(x)$ which satisfies the differential equation

$$\frac{dy}{dx} - \frac{y}{x} = x\cos x, \quad x > 0$$

Determine the general solution of this differential equation, and hence find the particular solution for which $y = \pi$ when $x = \frac{\pi}{2}$ **[3 marks]**

STRETCH

D5 **(a)** Showing all stages of your working, show that the substitution $z = y^{-2}$ transforms the differential equation $\frac{dy}{dx} - y = e^x y^3$ into the differential equation $\frac{dz}{dx} + 2z = -2e^x$

(2)

(b) By solving this differential equation, determine the general solution of the original equation, giving your answer in the form $y^2 = f(x)$

(2)

[4 marks]

D6 **(a)** Showing all stages of your working, show that the substitution $z = e^y$ transforms the differential equation $e^y \frac{dy}{dx} + e^y = x$ into the differential equation $\frac{dz}{dx} + z = x$

(2)

(b) Hence determine the particular solution of the original equation for which $y = 0$ when $x = 0$, giving your answer in the form $y = f(x)$

(2)

[4 marks]

D7 The population P of fish in a lake, t years after harvesting begins, is modelled by the differential equation

$$\frac{dP}{dt} = 0.1P - 20$$

Initially there are 150 fish in the lake.

(a) Determine P in terms of t

(2)

(b) According to the model, determine the exact time at which the fish population becomes zero, and comment on whether the model is suitable for large values of t

(2)

[4 marks]

D8 A family of curves satisfies the differential equation

$$\frac{dy}{dx} + \frac{2y}{x} = 6x, \quad x > 0$$

(a) Determine the general solution of this differential equation.

(2)

(b) Determine the particular solution whose curve passes through the point $(1, 2)$

(1)

(c) Using calculus, determine the exact coordinates of the minimum point of this particular solution curve, justifying that it is a minimum.

(2)

[5 marks]

E Exam practice

Exam-style questions in the style of WFM02. Write your answers in the spaces, and show every step.

CORE *Exam questions for everyone*

E1 A curve with equation $y = f(x)$ satisfies the differential equation

$$\frac{dy}{dx} + y\cot x = 2\cos x, \quad 0 < x < \pi$$

(a) Show that an integrating factor for this differential equation is given by $\sin x$

(2)

(b) Determine the general solution of the differential equation.

(2)

Given that $y = 1$ when $x = \frac{\pi}{2}$

(c) determine the particular solution of the differential equation, giving your answer in its simplest form $y = f(x)$

(2)

(Total for Question E1 is 6 marks)

E2 **(a)** Show that the substitution $z = \frac{1}{y}$ transforms the differential equation

$$x\frac{dy}{dx} + y = x^2y^2, \quad x > 0 \quad \text{(I)}$$

into the differential equation

$$\frac{dz}{dx} - \frac{z}{x} = -x \quad \text{(II)}$$

(3)

(b) By solving differential equation (II), determine the general solution of differential equation (I), giving your answer in the form $y = f(x)$, where $f(x)$ is a single fraction

(3)

(c) Determine the particular solution of (I) for which $y = \frac{1}{2}$ when $x = 1$

(1)

(Total for Question E2 is 7 marks)

STRETCH

E3 **(a)** Show that the substitution $z = y^2$ transforms the differential equation

$$2y\frac{dy}{dx} + y^2 \tan x = \sec x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \quad \text{(I)}$$

into the differential equation

$$\frac{dz}{dx} + z\tan x = \sec x \quad \text{(II)}$$

(3)

(b) By solving differential equation (II), determine the general solution of differential equation (I), giving your answer in the form $y^2 = f(x)$

(4)

Given that $y = 2$ when $x = 0$, and that $y > 0$,

(c) determine the exact value of y when $x = \frac{\pi}{4}$

(2)

(Total for Question E3 is 9 marks)

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