eClassroom

INTERNATIONAL A LEVEL MATHEMATICS • FURTHER PURE MATHEMATICS 3

# Further Pure 3

## Exam Questions by Topic: Mark Scheme

Pearson Edexcel International Advanced Level (WFM03/01) • Calculator allowed

Every question in the booklet, marked line by line the way an examiner marks it


| Chapters (tap a chapter to jump to it) | | Qs | Marks | Page | My score |
|---|---|---|---|---|---|
| 1 | **Hyperbolic Functions**<br>Definitions, graphs, inverse functions, identities and equations | 10 | 80 | 3 | |
| 2 | **Further Coordinate Systems**<br>Ellipses and hyperbolas, foci and directrices, tangents, normals and loci | 10 | 80 | 10 | |
| 3 | **Differentiation**<br>Hyperbolic and inverse trigonometric and hyperbolic functions | 10 | 80 | 17 | |
| 4 | **Integration**<br>Substitutions, reduction formulae, arc length and surface area | 10 | 80 | 23 | |
| 5 | **Vectors**<br>Vector product, triple products, lines and planes, distances and angles | 10 | 80 | 30 | |
| 6 | **Further Matrix Algebra**<br>$3 \times 3$ matrices, inverses, eigenvalues and eigenvectors, diagonalisation | 10 | 80 | 37 | |



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# How to use this mark scheme

The same marking rules Pearson examiners use for WFM03.

**Mark every line, not just the answer.** Method marks are given for a correct method even when the final answer is wrong. If you lost marks, find the *first* line where your working differs from the scheme. That is the thing to fix.

| Code | Meaning |
|---|---|
| **M1** | Method mark: for knowing a correct method and attempting to apply it. Small slips in the working do not lose this mark. |
| **dM1** | Dependent method mark: can only be given if the method mark before it has been given. |
| **A1** | Accuracy mark: for a correct answer or line of working. It can only be given if the method mark it depends on has been given. |
| **B1** | Independent mark: for a correct result or statement, whatever method was used. |
| * | The answer is given in the question. Every step must be shown, and the answer must follow from correct working (cso). |
| **cao** | Correct answer only. |
| **cso** | Correct solution only: no errors anywhere in the working. |
| **ft** | Follow through: the mark can be given for correct work using your own earlier (wrong) answer. |
| **oe** | Or equivalent: any correct equivalent form is accepted. |

**"Solutions relying entirely on calculator technology are not acceptable."** When a WFM03 question says this, a correct final answer with no algebraic working scores **no marks**. Exact answers (surds, multiples of $\pi$) are required when asked for; otherwise give 3 significant figures, with arguments in radians.

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# 1 Chapter 1: Hyperbolic Functions

Mark scheme. Marks are shown on the right of each line.

## Question 1

6 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $\frac{1}{4}\left(e^{x}+e^{-x}\right)^{2}-\frac{1}{4}\left(e^{x}-e^{-x}\right)^{2}$ | M1 |
| | $=\frac{1}{4}\left(e^{2x}+2+e^{-2x}-e^{2x}+2-e^{-2x}\right)$ | A1 |
| | $=1$ * | A1* |
| | *Summer 2014 report: at least one line of working is needed between the substitution and "= 1".* | (3) |
| (b) | $\sinh(\ln 3)=\frac{1}{2}\left(3-\frac{1}{3}\right)=\boxed{\frac{4}{3}}$ | B1 |
| | $\cosh(\ln 3)=\boxed{\frac{5}{3}}$ | B1 |
| | $\tanh(\ln 3)=\boxed{\frac{4}{5}}$ | B1 |
| | | (3) |
| | | Total for Question 1: 6 marks |

## Question 2

6 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $y=\operatorname{arcosh}x$: $x=\frac{1}{2}\left(e^{y}+e^{-y}\right)$ | M1 |
| | $e^{2y}-2xe^{y}+1=0$ | A1 |
| | $e^{y}=x\pm\sqrt{x^{2}-1}$ | M1 |
| | $y\geqslant 0$ needs the positive sign: $\operatorname{arcosh}x=\ln\left(x+\sqrt{x^{2}-1}\right)$ * | A1* |
| | | (4) |
| (b) | $\boxed{\ln\left(2+\sqrt{3}\right)}$ | B1 |
| | | (1) |
| (c) | $\frac{1}{2}\ln\frac{3/2}{1/2}=\boxed{\frac{1}{2}\ln 3}$ | B1 |
| | | (1) |
| | | Total for Question 2: 6 marks |

## Question 3

6 marks

| Part | Scheme | Marks |
|---|---|---|
| | $\frac{3}{2}\left(e^{x}+e^{-x}\right)+\frac{1}{2}\left(e^{x}-e^{-x}\right)=5$ | M1 |
| | $2e^{x}+e^{-x}=5$ | A1 |
| | $2e^{2x}-5e^{x}+1=0$ | M1 |
| | $e^{x}=\frac{5\pm\sqrt{17}}{4}$ | A1 |
| | Both are positive, so both give solutions | M1 |
| | $x=\ln\frac{5+\sqrt{17}}{4}$ or $x=\ln\frac{5-\sqrt{17}}{4}$ | A1 |
| | | (6) |
| | | **Total for Question 3: 6 marks** |

## Question 4

6 marks

| Part | Scheme | Marks |
|---|---|---|
| | $2\cosh^{2}x-1-5\cosh x+4=0$ | M1 |
| | $2\cosh^{2}x-5\cosh x+3=0$ | A1 |
| | $(2\cosh x-3)(\cosh x-1)=0$ | M1 |
| | $\cosh x=1$: $x=0$ | A1 |
| | $\cosh x=\frac{3}{2}$: $x=\pm\ln\left(\frac{3}{2}+\sqrt{\frac{5}{4}}\right)$ | M1 |
| | $x=0$, $x=\pm\ln\frac{3+\sqrt{5}}{2}$ | A1 |
| | *Both signs are needed, since* cosh *is even.* | |
| | | (6) |
| | | **Total for Question 4: 6 marks** |

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## Question 5

6 marks

| Part | Scheme | Marks |
|---|---|---|
| **(a)** | $\cosh x$: U-shape, minimum $(0, 1)$<br>$\sinh x$ through $O$, increasing, $\cosh x$ above it | **B1**<br>**B1**<br>**(2)** |
| **(b)** | $0 < \operatorname{sech} x \leqslant 1$<br>$\boxed{1 \leqslant f(x) < 3}$ | **M1**<br>**A1**<br>**(2)** |
| **(c)** | $\cosh x \geqslant 1$ for all real $x$ | **B1**<br>**(1)** |
| **(d)** | $\boxed{\text{Domain } x \geqslant 1, \text{ range } y \geqslant 0}$ | **B1**<br>**(1)** |
| | | **Total for Question 5: 6 marks** |

## Question 6

10 marks

| Part | Scheme | Marks |
|---|---|---|
| **(a)** | $\cosh^4 x = \frac{1}{16}\left(e^x + e^{-x}\right)^4$<br>$= \frac{1}{16}\left(e^{4x} + 4e^{2x} + 6 + 4e^{-2x} + e^{-4x}\right)$<br>$= \frac{1}{16}(2\cosh 4x + 8\cosh 2x + 6)$<br>$8\cosh^4 x = \cosh 4x + 4\cosh 2x + 3$ *<br>*Jan 2022 report: a significant number failed to use the exponential definitions at any stage.* | **M1**<br>**A1**<br>**M1**<br>**A1***<br>**(4)** |
| **(b)** | $8\cosh^4 x - 3 = 15$<br>$\cosh^4 x = \frac{9}{4}$<br>$\cosh^2 x = \frac{3}{2}$ (not $-\frac{3}{2}$)<br>$\cosh x = \sqrt{\frac{3}{2}}$<br>$x = \pm\ln\left(\sqrt{\frac{3}{2}} + \sqrt{\frac{1}{2}}\right)$<br>$\boxed{x = \pm\ln\frac{\sqrt{3}+1}{\sqrt{2}}}$<br>*Jan 2022 report: finding $\cosh x$ where $\cosh^2 x$ was needed lost the final marks.* | **M1**<br>**A1**<br>**A1**<br>**M1**<br>**M1**<br>**A1**<br>**(6)** |
| | | **Total for Question 6: 10 marks** |

## Question 7

10 marks

| Part | Scheme | Marks |
|---|---|---|
| **(a)** | $4\sinh x - 1 = \cosh x$ | **M1** |
| | $2\left(e^{x} - e^{-x}\right) - 1 = \frac{1}{2}\left(e^{x} + e^{-x}\right)$ | **M1** |
| | $3e^{2x} - 2e^{x} - 5 = 0$ | **A1** |
| | $(3e^{x} - 5)(e^{x} + 1) = 0$ | **M1** |
| | $e^{x} = -1$ is impossible | **A1** |
| | $\boxed{x = \ln\frac{5}{3}}$ | **A1** |
| | | **(6)** |
| **(b)** | $\cosh x = \frac{17}{15}$ gives $x = \pm\ln\frac{5}{3}$; squaring also accepts $4\sinh x = -(1+\cosh x)$ | **M1** |
| | $x = -\ln\frac{5}{3}$ is the extra solution: there $4\sinh x < 0 < 1 + \cosh x$ | **A1** |
| | | **(2)** |
| **(c)** | $\tanh x = \dfrac{\frac{5}{3} - \frac{3}{5}}{\frac{5}{3} + \frac{3}{5}} = \frac{8}{17}$, $\operatorname{sech} x = \frac{15}{17}$ | **M1** |
| | $\frac{32}{17} - \frac{15}{17} = 1$ * | **A1*** |
| | | **(2)** |
| | | **Total for Question 7: 10 marks** |

## Question 8

10 marks

| Part | Scheme | Marks |
|---|---|---|
| **(a)** | $y = \text{arsinh}\, x$: $x = \frac{1}{2}\left(e^{y} - e^{-y}\right)$ | **M1** |
| | $e^{2y} - 2xe^{y} - 1 = 0$ | **A1** |
| | $e^{y} = x \pm \sqrt{x^2 + 1}$ | **M1** |
| | $e^{y} > 0$, so the positive root: result * | **A1*** |
| | | **(4)** |
| **(b)** | $2\left(1 + \sinh^2 x\right) - \sinh x = 7$: $2\sinh^2 x - \sinh x - 5 = 0$ | **M1** |
| | $\sinh x = \dfrac{1 \pm \sqrt{41}}{4}$ | **A1** |
| | Both valid; uses the log form of arsinh | **M1** |
| | $x = \ln\left(\dfrac{1+\sqrt{41}}{4} + \sqrt{\dfrac{29+\sqrt{41}}{8}}\right)$ and $x = \ln\left(\dfrac{1-\sqrt{41}}{4} + \sqrt{\dfrac{29-\sqrt{41}}{8}}\right)$ | **A1** |
| | *Jan 2021 report: the logarithmic form of arsinh is quicker than starting again with exponentials.* | |
| | | **(4)** |
| **(c)** | $1.375$ | **B1** |
| | $-1.109$ | **B1** |
| | | **(2)** |
| | | **Total for Question 8: 10 marks** |

## Question 9

10 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $\tanh 2x = \dfrac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}$ | B1 |
| | RHS: $\dfrac{2\frac{e^x - e^{-x}}{e^x + e^{-x}}}{1 + \frac{(e^x - e^{-x})^2}{(e^x + e^{-x})^2}} = \dfrac{2(e^x - e^{-x})(e^x + e^{-x})}{(e^x + e^{-x})^2 + (e^x - e^{-x})^2}$ | M1 |
| | $= \dfrac{2\left(e^{2x} - e^{-2x}\right)}{2\left(e^{2x} + e^{-2x}\right)}$ | A1 |
| | $= \tanh 2x$ * | A1* |
| | | (4) |
| (b) | $\dfrac{2t}{1+t^2} = \frac{4}{5}$: $2t^2 - 5t + 2 = 0$ | M1 |
| | $t = \frac{1}{2}$ or $t = 2$ | A1 |
| | $t = 2$ rejected since $\lvert\tanh x\rvert < 1$ | M1 |
| | $x = \operatorname{artanh} \frac{1}{2} = \boxed{\frac{1}{2}\ln 3}$ | A1 |
| | | (4) |
| (c) | $\operatorname{artanh} \frac{4}{5} = \frac{1}{2}\ln 9 = \ln 3$ | M1 |
| | $2x = \ln 3$, so $x = \frac{1}{2}\ln 3$ * | A1* |
| | | (2) |
| | | **Total for Question 9: 10 marks** |

## Question 10

10 marks

| Part | Scheme | Marks |
|---|---|---|
| **(a)** | $R\cosh(x-\alpha) = R\cosh x\cosh\alpha - R\sinh x\sinh\alpha$ | **M1** |
| | $R\cosh\alpha = 5$, $R\sinh\alpha = 3$: $R^2 = 25 - 9$ | **M1** |
| | $\boxed{R = 4}$ | **A1** |
| | $\tanh\alpha = \frac{3}{5}$: $\boxed{\alpha = \frac{1}{2}\ln 4 = \ln 2}$ | **A1** |
| | | **(4)** |
| **(b)** | Minimum when $\cosh(x-\alpha) = 1$: value 4 | **B1** |
| | at $x = \ln 2$ | **B1** |
| | | **(2)** |
| **(c)** | $4\cosh(x - \ln 2) = 5$ | **M1** |
| | $x - \ln 2 = \pm\ln\left(\frac{5}{4} + \frac{3}{4}\right) = \pm\ln 2$ | **M1** |
| | Both signs | **A1** |
| | $x = 0$ or $x = \ln 4$ | **A1** |
| | | **(4)** |
| | | **Total for Question 10: 10 marks** |

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# 2 Chapter 2: Further Coordinate Systems

Mark scheme. Marks are shown on the right of each line.

## Question 1

6 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $16 = 25\left(1 - e^2\right)$ | M1 |
| | $\boxed{e = \frac{3}{5}}$ | A1 |
| | | (2) |
| (b) | $\boxed{\text{Foci } (\pm 3, 0)}$ | B1 |
| | $\boxed{\text{Directrices } x = \pm\frac{25}{3}}$ | B1 |
| | | (2) |
| (c) | $PS + PS' = e(PM + PM')$ | M1 |
| | $= \frac{3}{5} \times \frac{50}{3} = 10$ (the directrices are $\frac{50}{3}$ apart) * | A1* |
| | | (2) |
| | **Total for Question 1: 6 marks** | |

## Question 2

6 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $16 = 9\left(e^2 - 1\right)$ | M1 |
| | $\boxed{e = \frac{5}{3}}$ | A1 |
| | | (2) |
| (b) | $\boxed{\text{Foci } (\pm 5, 0)}$ | B1 |
| | $\boxed{\text{Directrices } x = \pm\frac{9}{5}}$ | B1 |
| | | (2) |
| (c) | $y = \pm\frac{4}{3}x$ | B1 |
| | $\boxed{y = \frac{4}{3}x \text{ and } y = -\frac{4}{3}x}$ | B1 |
| | | (2) |
| | **Total for Question 2: 6 marks** | |

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## Question 3

6 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $\frac{dy}{dx} = \frac{b\cos\theta}{-a\sin\theta}$ | M1 |
| | $y - b\sin\theta = -\frac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$ | M1 |
| | $ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$ | A1 |
| | $bx\cos\theta + ay\sin\theta = ab$ * | A1* |
| | | (4) |
| (b) | $3x \times \frac{1}{2} + 4y \times \frac{\sqrt{3}}{2} = 12$ | M1 |
| | $\boxed{3x + 4\sqrt{3}y = 24}$ | A1 |
| | | (2) |
| | | Total for Question 3: 6 marks |

## Question 4

6 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $\frac{dy}{dx} = \frac{3\cosh t}{4\sinh t}$ | M1 |
| | Normal gradient $-\frac{4\sinh t}{3\cosh t}$ | M1 |
| | $y - 3\sinh t = -\frac{4\sinh t}{3\cosh t}(x - 4\cosh t)$ | M1 |
| | $3y\cosh t - 9\sinh t\cosh t = -4x\sinh t + 16\sinh t\cosh t$ | A1 |
| | $4x\sinh t + 3y\cosh t = 25\sinh t\cosh t$ * | A1* |
| | | (5) |
| (b) | $\boxed{x = \frac{25}{4}\cosh t}$ | B1 |
| | | (1) |
| | | Total for Question 4: 6 marks |

## Question 5

6 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $\sqrt{(x-4)^2+y^2} = \frac{2}{3}\lvert x-9\rvert$ | M1 |
| | $9\left((x-4)^2+y^2\right) = 4(x-9)^2$ | M1 |
| | $9x^2-72x+144+9y^2 = 4x^2-72x+324$ | A1 |
| | $5x^2+9y^2 = 180$ | A1 |
| | $\frac{x^2}{36}+\frac{y^2}{20} = 1$ * | A1* |
| | | (5) |
| (b) | $20 = 36\left(1-e^2\right)$ gives $e^2 = \frac{4}{9}$, $e = \frac{2}{3}$ * | B1 |
| | | (1) |
| | | **Total for Question 5: 6 marks** |

## Question 6

10 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $4 = 9\left(1-e^2\right)$ | M1 |
| | $e = \frac{\sqrt{5}}{3}$ (positive, and less than 1) | A1 |
| | Foci $\left(\pm\sqrt{5}, 0\right)$ | B1 |
| | Directrices $x = \pm\frac{9}{\sqrt{5}}$ | B1 |
| | *Jan 2022 report: some rejected $e = \frac{\sqrt{5}}{3}$, not realising that $0.745 < 1$.* | (4) |
| (b) | $\frac{dy}{dx} = -\frac{2\cos\theta}{3\sin\theta}$ | M1 |
| | $y - 2\sin\theta = -\frac{2\cos\theta}{3\sin\theta}(x - 3\cos\theta)$ | M1 |
| | $2x\cos\theta + 3y\sin\theta = 6$ * | A1* |
| | | (3) |
| (c) | $l$: $y = \frac{3\sin\theta}{2\cos\theta}x$ | B1 |
| | $2x\cos\theta + \frac{9\sin^2\theta}{2\cos\theta}x = 6$ | M1 |
| | $x = \frac{12\cos\theta}{4\cos^2\theta+9\sin^2\theta}$ | A1 |
| | *Jan 2022 report: the most common error was the missing $x$ in the equation of $l$.* | (3) |
| | | **Total for Question 6: 10 marks** |

## Question 7

10 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $\frac{dy}{dx} = \frac{\sec^2\theta}{2\sec\theta\tan\theta} = \frac{\sec\theta}{2\tan\theta}$ | M1 |
| | $y - \tan\theta = \frac{\sec\theta}{2\tan\theta}(x - 2\sec\theta)$ | M1 |
| | $2y\tan\theta - 2\tan^2\theta = x\sec\theta - 2\sec^2\theta$ | A1 |
| | Uses $\sec^2\theta - \tan^2\theta = 1$: $x\sec\theta - 2y\tan\theta = 2$ * | A1* |
| | | (4) |
| (b) | Asymptotes $y = \pm\frac{1}{2}x$ | B1 |
| | $A$: $x = \frac{2}{\sec\theta - \tan\theta}$; $B$: $x = \frac{2}{\sec\theta + \tan\theta}$ | M1 |
| | Mean of $x$: $\frac{2\sec\theta}{\sec^2\theta - \tan^2\theta} = 2\sec\theta$ | A1 |
| | Mean of $y$: $\tan\theta$, so $P$ is the midpoint * | A1* |
| | | (4) |
| (c) | Area $= \frac{1}{2}\lvert x_A y_B - x_B y_A\rvert = \frac{1}{2} \times \frac{4}{\sec^2\theta - \tan^2\theta}$ | M1 |
| | $\boxed{2}$, independent of $\theta$ | A1 |
| | | (2) |
| | | **Total for Question 7: 10 marks** |

## Question 8

10 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | Tangent gradient $-\frac{3\cos\theta}{4\sin\theta}$, normal gradient $\frac{4\sin\theta}{3\cos\theta}$ | M1 |
| | $y-3\sin\theta=\frac{4\sin\theta}{3\cos\theta}(x-4\cos\theta)$ | M1 |
| | $3y\cos\theta-9\sin\theta\cos\theta=4x\sin\theta-16\sin\theta\cos\theta$ | A1 |
| | Divides by $\sin\theta\cos\theta$: $\frac{4x}{\cos\theta}-\frac{3y}{\sin\theta}=7$ * | A1* |
| | | (4) |
| (b) | $\boxed{A\left(\frac{7}{4}\cos\theta,0\right)}$ | B1 |
| | $\boxed{B\left(0,-\frac{7}{3}\sin\theta\right)}$ | B1 |
| | | (2) |
| (c) | $M\left(\frac{7}{8}\cos\theta,-\frac{7}{6}\sin\theta\right)$ | M1 |
| | $\cos\theta=\frac{8x}{7}$, $\sin\theta=-\frac{6y}{7}$ | A1 |
| | $\left(\frac{8x}{7}\right)^2+\left(\frac{6y}{7}\right)^2=1$ | M1 |
| | $64x^2+36y^2=49$ * | A1* |
| | | (4) |
| | | **Total for Question 8: 10 marks** |

## Question 9

10 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $9 = 16\left(e^2 - 1\right)$: $\boxed{e = \frac{5}{4}}$ | B1 |
| | $\boxed{S(5,0),\ S'(-5,0)}$ | B1 |
| | $\boxed{x = \pm\frac{16}{5}}$ | B1 |
| | | (3) |
| (b) | $PS = e\left(8 - \frac{16}{5}\right)$ | M1 |
| | $\boxed{6}$ | A1 |
| | $P\left(8, 3\sqrt{3}\right)$: $\sqrt{3^2 + 27} = 6$ | B1 |
| | | (3) |
| (c) | $PS = e\left(x - \frac{a}{e}\right) = ex - a$ | M1 |
| | $PS' = e\left(x + \frac{a}{e}\right) = ex + a$ | M1 |
| | $PS' - PS = 2a$ | A1 |
| | $= 8$ * | A1* |
| | | (4) |
| | | **Total for Question 9: 10 marks** |

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## Question 10

10 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $\frac{x^2}{25}+\frac{y^2}{9}=1: 9=25\left(1-e^2\right)$, $\boxed{e=\frac{4}{5}}$ | B1 |
| | $\boxed{S(4,0)}$ | B1 |
| | $\boxed{x=\frac{25}{4}}$ | B1 |
| | | (3) |
| (b) | $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{3\cos\theta}{5\sin\theta}$ | M1 |
| | $y-3\sin\theta=-\frac{3\cos\theta}{5\sin\theta}(x-5\cos\theta)$ | M1 |
| | $3x\cos\theta+5y\sin\theta=15$ * | A1* |
| | | (3) |
| (c) | $OT=\frac{5}{\cos\theta}$, $ON=5\cos\theta$ | M1 |
| | Product = 25 * | A1* |
| | | (2) |
| (d) | $D\left(\frac{25}{4}, \frac{3-\frac{15}{4}\cos\theta}{\sin\theta}\right)$; $\overrightarrow{SP}\cdot\overrightarrow{SD}=(5\cos\theta-4)\times\frac{9}{4}+3\left(3-\frac{15}{4}\cos\theta\right)$ | M1 |
| | $=\frac{45}{4}\cos\theta-9+9-\frac{45}{4}\cos\theta=0$, so angle $PSD=90^\circ$ * | A1* |
| | | (2) |
| | | **Total for Question 10: 10 marks** |

# 3 Chapter 3: Differentiation

Mark scheme. Marks are shown on the right of each line.

## Question 1

6 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | Product rule<br>$\boxed{3\sinh 3x\sinh 2x+2\cosh 3x\cosh 2x}$ | M1<br>A1<br>(2) |
| (b) | $2\tanh 2x\times \operatorname{sech}^2 2x\times 2$<br>$\boxed{4\tanh 2x\operatorname{sech}^2 2x}$<br>*Jan 2021 report: the factor 2 from the inner function was often omitted.* | M1<br>A1<br>(2) |
| (c) | $\boxed{\tanh x}$ | B1<br>(1) |
| (d) | $\boxed{-\operatorname{sech} x\tanh x}$ | B1<br>(1) |
| | | **Total for Question 1: 6 marks** |

## Question 2

6 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $x=\sin y,\ \frac{\mathrm{d}x}{\mathrm{d}y}=\cos y$<br>$\cos y=\sqrt{1-\sin^2 y}=\sqrt{1-x^2}$ (positive for $-\frac{\pi}{2}<y<\frac{\pi}{2}$)<br>$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\sqrt{1-x^2}}$ * | M1<br>M1<br>A1*<br>(3) |
| (b) | $\boxed{\frac{3}{1+9x^2}}$ | B1<br>(1) |
| (c) | $-\frac{1}{\sqrt{1-x^4}}\times 2x$<br>$\boxed{-\frac{2x}{\sqrt{1-x^4}}}$ | M1<br>A1<br>(2) |
| | | **Total for Question 2: 6 marks** |

## Question 3

6 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $x = \tanh y,\ \frac{dx}{dy} = \operatorname{sech}^2 y$ | M1 |
| | $= 1 - \tanh^2 y = 1 - x^2$ | M1 |
| | $\frac{dy}{dx} = \frac{1}{1-x^2}$ * | A1* |
| | | (3) |
| (b) | $\boxed{\frac{2}{\sqrt{4x^2+1}}}$ | B1 |
| | | (1) |
| (c) | Product rule | M1 |
| | $\boxed{\operatorname{arcosh} x + \frac{x}{\sqrt{x^2-1}}}$ | A1 |
| | | (2) |
| | | Total for Question 3: 6 marks |

## Question 4

6 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $\frac{dy}{dx} = 2\sinh x + \cosh x = 0$ | M1 |
| | $\tanh x = -\frac{1}{2},\ x = -\frac{1}{2}\ln 3$ | A1 |
| | $\cosh x = \frac{1}{\sqrt{1-\frac{1}{4}}} = \frac{2}{\sqrt{3}},\ \sinh x = -\frac{1}{\sqrt{3}}$ | M1 |
| | $\boxed{\left(-\frac{1}{2}\ln 3,\ \sqrt{3}\right)}$ | A1 |
| | | (4) |
| (b) | $\frac{d^2y}{dx^2} = 2\cosh x + \sinh x = y = \sqrt{3} > 0$ | M1 |
| | $\boxed{\text{Minimum}}$ | A1 |
| | | (2) |
| | | Total for Question 4: 6 marks |

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## Question 5

6 marks

| Part | Scheme | Marks |
|---|---|---|
| | $y' = (1-x^2)^{-\frac{1}{2}}$ | B1 |
| | $y'' = -\frac{1}{2}(1-x^2)^{-\frac{3}{2}}(-2x)$ | M1 |
| | $= x(1-x^2)^{-\frac{3}{2}}$ | A1 |
| | $(1-x^2)y'' = x(1-x^2)^{-\frac{1}{2}}$ | M1 |
| | $= xy'$ | A1 |
| | so $(1-x^2)y'' = xy'$ * | A1* |
| | | (6) |
| | | **Total for Question 5: 6 marks** |

## Question 6

10 marks

| Part | Scheme | Marks |
|---|---|---|
| **(a)** | $\operatorname{sech} y = \frac{x}{2}$ | M1 |
| | $-\operatorname{sech} y \tanh y \frac{dy}{dx} = \frac{1}{2}$ (with the minus sign) | A1 |
| | $\tanh y = \sqrt{1-\operatorname{sech}^2 y} = \sqrt{1-\frac{x^2}{4}}$ | M1 |
| | $\frac{dy}{dx} = -\frac{1}{2\cdot\frac{x}{2}\sqrt{1-\frac{x^2}{4}}}$ | A1 |
| | $= -\frac{2}{x\sqrt{4-x^2}}$ * | A1* |
| | *Jan 2022 report: omitting the minus sign when differentiating* $\operatorname{sech} y$ *was a common slip.* | (5) |
| **(b)** | $\frac{1}{1-x^2} = \frac{4}{3} : x^2 = \frac{1}{4}$ | M1 |
| | $\boxed{x = \frac{1}{2}}$ ($-\frac{1}{2}$ is outside $0 < x < 1$) | A1 |
| | | (2) |
| **(c)** | $y = \operatorname{arsech}\frac{1}{2} = \operatorname{arcosh} 2 = \ln(2+\sqrt{3})$ | B1 |
| | Gradient $-\frac{2}{\sqrt{3}}$ | M1 |
| | $\boxed{y = -\frac{2}{\sqrt{3}}x + \frac{2}{\sqrt{3}} + \ln(2+\sqrt{3})}$ | A1 |
| | | (3) |
| | | **Total for Question 6: 10 marks** |

## Question 7

10 marks

| Part | Scheme | Marks |
|---|---|---|
| **(a)** | Product rule: $-e^{-x}\cosh 2x + e^{-x} \times 2\sinh 2x$<br>$= e^{-x}(2\sinh 2x - \cosh 2x)$ * | **M1**<br>**A1***<br>**(2)** |
| **(b)** | $e^{-x} \neq 0$, so $\tanh 2x = \frac{1}{2}$<br>$2x = \frac{1}{2}\ln\frac{1+\frac{1}{2}}{1-\frac{1}{2}} = \frac{1}{2}\ln 3$<br>$x = \frac{1}{4}\ln 3$ * | **M1**<br>**M1**<br>**A1***<br>**(3)** |
| **(c)** | $\cosh 2x = \frac{1}{\sqrt{1-\frac{1}{4}}} = \frac{2}{\sqrt{3}}$<br>$e^{-x} = 3^{-\frac{1}{4}}$<br>$\boxed{y = 2 \times 3^{-\frac{3}{4}}}$ | **M1**<br>**M1**<br>**A1**<br>**(3)** |
| **(d)** | Sign of $2\sinh 2x - \cosh 2x$: negative for $\tanh 2x < \frac{1}{2}$, positive for $\tanh 2x > \frac{1}{2}$<br>Minimum | **M1**<br>**A1**<br>**(2)** |
| | | **Total for Question 7: 10 marks** |

## Question 8

10 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $\frac{1}{1+x^2}+\frac{1}{1+\frac{1}{x^2}}\times\left(-\frac{1}{x^2}\right)$ | M1 |
| | $=\frac{1}{1+x^2}-\frac{1}{x^2+1}=0$ | A1 |
| | So the expression is constant; at $x=1$ it is $\frac{\pi}{4}+\frac{\pi}{4}$ | M1 |
| | $\boxed{\frac{\pi}{2}}$ | A1 |
| | | (4) |
| (b) | $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2x}{1+x^2}\right)=\frac{2(1-x^2)}{(1+x^2)^2}$ | B1 |
| | $1-\left(\frac{2x}{1+x^2}\right)^2=\frac{(1-x^2)^2}{(1+x^2)^2}$ | M1 |
| | $\frac{(1+x^2)}{1-x^2}\times\frac{2(1-x^2)}{(1+x^2)^2}$ (using $1-x^2>0$) | M1 |
| | $=\frac{2}{1+x^2}$ * | A1* |
| | | (4) |
| (c) | Both sides have the same derivative, so they differ by a constant | M1 |
| | At $x=0$ both are 0, so they are equal * | A1* |
| | | (2) |
| | | **Total for Question 8: 10 marks** |

## Question 9

10 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $y' = \left(1+x^2\right)^{-\frac{1}{2}}$ | B1 |
| | $y'' = -x\left(1+x^2\right)^{-\frac{3}{2}}$ | M1 |
| | Correct | A1 |
| | $\left(1+x^2\right)y'' = -x\left(1+x^2\right)^{-\frac{1}{2}} = -xy'$ | M1 |
| | Result * | A1* |
| | | (5) |
| (b) | $y' = \dfrac{2\operatorname{arsinh}x}{\sqrt{1+x^2}}$, so $\sqrt{1+x^2}\,y' = 2\operatorname{arsinh}x$ | M1 |
| | Differentiates both sides: $\dfrac{x}{\sqrt{1+x^2}}y' + \sqrt{1+x^2}\,y'' = \dfrac{2}{\sqrt{1+x^2}}$ | M1 |
| | Correct | A1 |
| | Multiplies by $\sqrt{1+x^2}$ | M1 |
| | $\left(1+x^2\right)y'' + xy' = 2$ * | A1* |
| | | (5) |
| | | **Total for Question 9: 10 marks** |

## Question 10

10 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $\mathrm{f}'(x) = \dfrac{2}{\sqrt{x^2-1}} - 1$ | B1 |
| | $= 0$: $\sqrt{x^2-1} = 2$ | M1 |
| | $x = \sqrt{5}$ (positive, $x \geqslant 1$) | A1 |
| | $y = 2\ln\left(\sqrt{5}+2\right) - \sqrt{5}$: $\boxed{\left(\sqrt{5},\ 2\ln\left(2+\sqrt{5}\right) - \sqrt{5}\right)}$ | A1 |
| | $\mathrm{f}''(x) = -\dfrac{2x}{(x^2-1)^{\frac{3}{2}}}$ | M1 |
| | $\mathrm{f}''\left(\sqrt{5}\right) < 0$: maximum * | A1* |
| | | (6) |
| (b) | $\mathrm{f}'\left(\frac{5}{3}\right) = \dfrac{2}{\frac{4}{3}} - 1 = \frac{1}{2}$ | M1 |
| | $\operatorname{arcosh}\frac{5}{3} = \ln\left(\frac{5}{3}+\frac{4}{3}\right) = \ln 3$ | B1 |
| | $y - \left(2\ln 3 - \frac{5}{3}\right) = \frac{1}{2}\left(x - \frac{5}{3}\right)$ | M1 |
| | $\boxed{y = \frac{1}{2}x + 2\ln 3 - \frac{5}{2}}$ | A1 |
| | | (4) |
| | | **Total for Question 10: 10 marks** |

# 4 Chapter 4: Integration

Mark scheme. Marks are shown on the right of each line.

## Question 1

6 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $\frac{1}{2}\int(\cosh 2x+1)\,\mathrm{d}x$ | M1 |
| | $\boxed{\frac{1}{4}\sinh 2x+\frac{1}{2}x+c}$ | A1 |
| | | (2) |
| (b) | $\cosh(\ln 2)-1=\frac{5}{4}-1$ | M1 |
| | $\boxed{\frac{1}{4}}$ | A1 |
| | | (2) |
| (c) | $\int\frac{\sinh x}{\cosh x}\,\mathrm{d}x$ | M1 |
| | $\boxed{\ln\cosh x+c}$ | A1 |
| | | (2) |
| | **Total for Question 1: 6 marks** | |

## Question 2

6 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $\left[\operatorname{arsinh}\frac{x}{3}\right]_0^3=\operatorname{arsinh}1$ | M1 |
| | $\boxed{\ln\left(1+\sqrt{2}\right)}$ | A1 |
| | | (2) |
| (b) | $\left[\frac{1}{2}\arctan\frac{x}{2}\right]_0^2$ | M1 |
| | $\boxed{\frac{\pi}{8}}$ | A1 |
| | | (2) |
| (c) | $\operatorname{arcosh}\frac{5}{3}-\operatorname{arcosh}1=\ln\left(\frac{5}{3}+\frac{4}{3}\right)-0$ | M1 |
| | $\boxed{\ln 3}$ | A1 |
| | | (2) |
| | **Total for Question 2: 6 marks** | |

## Question 3

6 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $(x+2)^2+9$ | B1 |
| | $\text{arsinh}\left(\frac{x+2}{3}\right)$ | M1 |
| | $\boxed{\text{arsinh}\left(\frac{x+2}{3}\right)+c}$ | A1 |
| | | (3) |
| (b) | $9-(x-1)^2$ | B1 |
| | $\arcsin\left(\frac{x-1}{3}\right)$ | M1 |
| | $\boxed{\arcsin\left(\frac{x-1}{3}\right)+c}$ | A1 |
| | *Jan 2022 report: a minus sign cannot be extracted from the square root.* | |
| | | (3) |
| | | **Total for Question 3: 6 marks** |

## Question 4

6 marks

| Part | Scheme | Marks |
|---|---|---|
| | $dx = 2\cosh u\,du;\ \sqrt{4+4\sinh^2 u} = 2\cosh u$ | M1 |
| | $\int 4\cosh^2 u\,du$, limits 0 to arsinh 1 | A1 |
| | $2\int(\cosh 2u+1)\,du = [\sinh 2u+2u]$ | M1 |
| | $=[2\sinh u\cosh u+2u]_0^{\text{arsinh}1}$ | A1 |
| | $\sinh u = 1,\ \cosh u = \sqrt{2}$ | M1 |
| | $\boxed{2\ln\left(1+\sqrt{2}\right)+2\sqrt{2}}$ | A1 |
| | | (6) |
| | | **Total for Question 4: 6 marks** |

## Question 5

6 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $\sqrt{1+\sinh^2 x} = \cosh x$ | M1 |
| | $[\sinh x]_0^{\ln 3} = \sinh(\ln 3)$ | M1 |
| | $\boxed{\frac{4}{3}}$ | A1 |
| | | (3) |
| (b) | $y' = x^{\frac{1}{2}}$, $\sqrt{1+x}$ | M1 |
| | $\left[\frac{2}{3}(1+x)^{\frac{3}{2}}\right]_0^3 = \frac{2}{3}(8-1)$ | M1 |
| | $\boxed{\frac{14}{3}}$ | A1 |
| | | (3) |
| | | **Total for Question 5: 6 marks** |

## Question 6

10 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $I_n = [-x^n \cos x]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} nx^{n-1}\cos x \, dx$ | M1 |
| | First term is 0 | A1 |
| | Parts again: $n\left[x^{n-1}\sin x\right]_0^{\frac{\pi}{2}} - n(n-1)\int x^{n-2}\sin x \, dx$ | M1 |
| | $= n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}$ | A1 |
| | Complete, with brackets correct * | A1* |
| | *Jan 2022 report: working backwards to "fudge" the given answer earns no credit.* | |
| | | (5) |
| (b) | $I_0 = \int_0^{\frac{\pi}{2}} \sin x \, dx = 1$ | B1 |
| | $I_2 = 2\left(\frac{\pi}{2}\right) - 2I_0 = \pi - 2$ | M1 |
| | Correct | A1 |
| | $I_4 = 4\left(\frac{\pi}{2}\right)^3 - 12(\pi - 2)$ | M1 |
| | $\boxed{\frac{1}{2}\pi^3 - 12\pi + 24}$ | A1 |
| | | (5) |
| | | **Total for Question 6: 10 marks** |

## Question 7

10 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $\dot{x} = 2t$, $\dot{y} = 2$; $\sqrt{4t^2+4} = 2\sqrt{t^2+1}$ | M1 |
| | $s = \int_0^1 2\sqrt{t^2+1}\,\mathrm{d}t$ | A1 |
| | $t = \sinh u$: $2\int \cosh^2 u\,\mathrm{d}u = [\sinh u \cosh u + u]$ | M1 |
| | Limits 0 to arsinh 1: $\sqrt{2} + \operatorname{arsinh} 1$ | M1 |
| | $= \sqrt{2} + \ln\left(1+\sqrt{2}\right)$ * | A1* |
| | | (5) |
| (b) | $S = 2\pi\int_0^1 y\sqrt{\dot{x}^2+\dot{y}^2}\,\mathrm{d}t$ | M1 |
| | $= 2\pi\int_0^1 2t \times 2\sqrt{t^2+1}\,\mathrm{d}t$ | A1 |
| | $= 8\pi\left[\frac{1}{3}\left(t^2+1\right)^{\frac{3}{2}}\right]_0^1$ | M1 |
| | $\frac{8\pi}{3}\left(2\sqrt{2}-1\right)$ | A1 |
| | $\boxed{\frac{8\pi}{3}\left(2\sqrt{2}-1\right)}$ | A1 |
| | *Jan 2022 report: the $y$ in the formula was sometimes missing or not replaced.* | |
| | | (5) |
| | | **Total for Question 7: 10 marks** |

## Question 8

10 marks

| Part | Scheme | Marks |
|---|---|---|
| **(a)** | $\mathrm{d}x = 4\sinh\theta\,\mathrm{d}\theta$; $\sqrt{16\cosh^2\theta - 16} = 4\sinh\theta$ | **M1** |
| | Limits: $\cosh\theta = 1$ gives 0; $\cosh\theta = \frac{5}{4}$ gives $\ln 2$ | **B1** |
| | $16\int_0^{\ln 2} \sinh^2\theta\,\mathrm{d}\theta$ * | **A1*** |
| | *Jan 2021 report: simplifying the square root after the substitution was the main difficulty.* | **(3)** |
| **(b)** | $8\int(\cosh 2\theta - 1)\,\mathrm{d}\theta$ | **M1** |
| | $8[\sinh\theta\cosh\theta - \theta]_0^{\ln 2}$ | **A1** |
| | $\sinh(\ln 2) = \frac{3}{4}$, $\cosh(\ln 2) = \frac{5}{4}$ | **M1** |
| | $8\left(\frac{15}{16} - \ln 2\right)$ | **A1** |
| | $= \frac{15}{2} - 8\ln 2$ * | **A1*** |
| | | **(5)** |
| **(c)** | The area between $y = \sqrt{x^2 - 16}$, the $x$-axis and $x = 5$ | **B1** |
| | $I \approx 1.955$: the curve rises from $(4, 0)$ to $(5, 3)$ and lies above the chord, so the area lies between the triangle (1.5) and the rectangle (3): reasonable | **B1** |
| | | **(2)** |
| | | **Total for Question 8: 10 marks** |

## Question 9

10 marks

| Part | Scheme | Marks |
|---|---|---|
| **(a)** | $63+4x-4x^2=64-(2x-1)^2$ | **M1** |
| | Correct (not $62-\ldots$) | **A1** |
| | $\int \frac{\mathrm{d}x}{\sqrt{64-(2x-1)^2}}$; $u=2x-1$, $\mathrm{d}x=\frac{1}{2}\mathrm{d}u$ | **M1** |
| | $\frac{1}{2}\arcsin\frac{u}{8}$ | **A1** |
| | $\boxed{\frac{1}{2}\arcsin\left(\frac{2x-1}{8}\right)+c}$ | **A1** |
| | *Jan 2022 report: the multiplier $\frac{1}{2}$ was often lost.* | **(5)** |
| **(b)** | $4x^2+4x+5=(2x+1)^2+4$ | **M1** |
| | $\int \frac{\mathrm{d}x}{(2x+1)^2+4}=\frac{1}{2}\times\frac{1}{2}\arctan\frac{2x+1}{2}$ | **M1** |
| | $\left[\frac{1}{4}\arctan\frac{2x+1}{2}\right]_{-\frac{1}{2}}^{\frac{1}{2}}$ | **A1** |
| | $\frac{1}{4}(\arctan 1-\arctan 0)$ | **M1** |
| | $\boxed{\frac{\pi}{16}}$ | **A1** |
| | | **(5)** |
| | | **Total for Question 9: 10 marks** |

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## Question 10

10 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $y' = \sinh\frac{x}{2}$; $\sqrt{1+\sinh^2\frac{x}{2}} = \cosh\frac{x}{2}$ | M1 |
| | $\left[2\sinh\frac{x}{2}\right]_0^{2\ln 3} = 2\sinh(\ln 3)$ | M1 |
| | $\boxed{\frac{8}{3}}$ | A1 |
| | | (3) |
| (b) | $S = 2\pi\int 2\cosh\frac{x}{2}\times\cosh\frac{x}{2}\,dx$ | M1 |
| | $= 2\pi\int_0^{2\ln 3}(1+\cosh x)\,dx$ | A1 |
| | $= 2\pi[x+\sinh x]_0^{2\ln 3}$ | M1 |
| | $\sinh(2\ln 3) = \frac{1}{2}\left(9-\frac{1}{9}\right) = \frac{40}{9}$ | A1 |
| | $4\pi\ln 3 + \frac{80\pi}{9}$ * | A1* |
| | | (5) |
| (c) | End discs: radius $2\cosh(\ln 3) = \frac{10}{3}$ and radius 2: $\frac{100\pi}{9} + 4\pi$ | M1 |
| | $\boxed{4\pi\ln 3 + 24\pi}$ | A1 |
| | *Jan 2022 report: only a small number added the end circles when "total" surface area was asked for.* | |
| | | (2) |
| | **Total for Question 10: 10 marks** | |

# 5 Chapter 5: Vectors

Mark scheme. Marks are shown on the right of each line.

## Question 1

6 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | Determinant method | M1 |
| | $\boxed{4\mathbf{i}-5\mathbf{j}-6\mathbf{k}}$ | A1 |
| | | (2) |
| (b) | $4-10+6=0$ * | B1 |
| | | (1) |
| (c) | $\frac{1}{2}\sqrt{16+25+36}$ | M1 |
| | $\boxed{\frac{1}{2}\sqrt{77}}$ | A1 |
| | | (2) |
| (d) | $\boxed{\frac{1}{\sqrt{77}}(4\mathbf{i}-5\mathbf{j}-6\mathbf{k})}$ | B1 |
| | | (1) |
| | | **Total for Question 1: 6 marks** |

## Question 2

6 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $\overrightarrow{OB}\times\overrightarrow{OC}=\mathbf{i}-2\mathbf{j}+6\mathbf{k}$ | M1 |
| | Correct | A1 |
| | $\overrightarrow{OA}\cdot(\ldots)=1+12=13$ | M1 |
| | Volume $=\frac{1}{6}\times 13=\boxed{\frac{13}{6}}$ | A1 |
| | | (4) |
| (b) | The volume would be zero | B1 |
| | so $O$, $A$, $B$, $C$ would lie in one plane (coplanar) | B1 |
| | | (2) |
| | | **Total for Question 2: 6 marks** |

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## Question 3

6 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $\overrightarrow{AB} = 2\mathbf{i}+\mathbf{j}-\mathbf{k}$, $\overrightarrow{AC} = \mathbf{i}-\mathbf{j}+2\mathbf{k}$ | B1 |
| | $\mathbf{n} = \overrightarrow{AB}\times\overrightarrow{AC} = \mathbf{i}-5\mathbf{j}-3\mathbf{k}$ | M1 |
| | $\mathbf{n}\cdot\mathbf{a} = 1-6 = -5$ | M1 |
| | $\boxed{x-5y-3z=-5}$ | A1 |
| | | (4) |
| (b) | $2+5-12=-5$ * | B1 |
| | | (1) |
| (c) | $\boxed{\mathbf{r}\cdot(\mathbf{i}-5\mathbf{j}-3\mathbf{k})=-5}$ | B1 |
| | | (1) |
| | | **Total for Question 3: 6 marks** |

## Question 4

6 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $(1+2\lambda)+2(2-\lambda)+2(3\lambda)=13$ | M1 |
| | $\lambda = \frac{4}{3}$ | A1 |
| | $\boxed{\left(\frac{11}{3},\frac{2}{3},4\right)}$ | A1 |
| | | (3) |
| (b) | $\mathbf{b}\cdot\mathbf{n} = 2-2+6=6$ | M1 |
| | $\sin\theta = \dfrac{6}{\sqrt{14}\times 3}$ (angle with the plane, not with the normal) | M1 |
| | $\boxed{32.3^\circ}$ | A1 |
| | *Jan 2022 report: many were unclear which angle they had found and did not adjust it.* | |
| | | (3) |
| | | **Total for Question 4: 6 marks** |

## Question 5

6 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $\frac{\lvert 2(3)-(-1)+2(2)-1\rvert}{\sqrt{4+1+4}}$ | M1 |
| | $=\frac{10}{3}$ | A1 |
| | $\boxed{\frac{10}{3}}$ | A1 |
| | | (3) |
| (b) | $\mathbf{r}=\mathbf{p}+t(2\mathbf{i}-\mathbf{j}+2\mathbf{k})$ into the plane: $11+9t=1$ | M1 |
| | $t=-\frac{10}{9}$ | A1 |
| | $\boxed{\left(\frac{7}{9},\frac{1}{9},-\frac{2}{9}\right)}$ | A1 |
| | | (3) |
| | | **Total for Question 5: 6 marks** |

## Question 6

10 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $(\mathbf{i}+2\mathbf{j}+2\mathbf{k})\cdot(2\mathbf{i}-2\mathbf{j}+\mathbf{k})=2-4+2=0$ | M1 |
| | and $A$ is not on $\Pi$ $(4-2-1=1\neq 9)$, so $l$ is parallel to $\Pi$ * | A1* |
| | Distance $=\frac{\lvert 1-9\rvert}{3}$ | M1 |
| | $\frac{8}{3}$ | A1 |
| | $\boxed{\frac{8}{3}}$ | A1 |
| | | (5) |
| (b) | $\mathbf{a}+t\mathbf{n}$ in the plane: $1+9t=9$, $t=\frac{8}{9}$ (foot) | M1 |
| | Reflection uses $2t=\frac{16}{9}$ | M1 |
| | $\boxed{A'\left(\frac{50}{9},-\frac{23}{9},\frac{7}{9}\right)}$ | A1 |
| | | (3) |
| (c) | Same direction through $A'$ | M1 |
| | $\boxed{\mathbf{r}=\frac{50}{9}\mathbf{i}-\frac{23}{9}\mathbf{j}+\frac{7}{9}\mathbf{k}+\mu(\mathbf{i}+2\mathbf{j}+2\mathbf{k})}$ | A1 |
| | *Jan 2022 report: give the answer as "$\mathbf{r}=\ldots$".* | |
| | | (2) |
| | | **Total for Question 6: 10 marks** |

## Question 7

10 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $\mathbf{b}_1 \times \mathbf{b}_2 = \mathbf{i} - \mathbf{j} - 2\mathbf{k}$ | M1 |
| | Correct | A1 |
| | $\mathbf{a}_2 - \mathbf{a}_1 = -\mathbf{i} + 3\mathbf{j} - \mathbf{k}$; dot product $-1 - 3 + 2 = -2$ | M1 |
| | $\frac{\lvert -2 \rvert}{\sqrt{6}}$ | M1 |
| | $\boxed{\frac{\sqrt{6}}{3}}$ | A1 |
| | | (5) |
| (b) | The shortest distance is not zero | B1 |
| | so there is no common point; the lines are not parallel either, so they are skew | B1 |
| | | (2) |
| (c) | $\cos\theta = \frac{2}{\sqrt{2}\sqrt{5}}$ | M1 |
| | Correct | A1 |
| | $\boxed{50.8°}$ | A1 |
| | | (3) |
| | **Total for Question 7: 10 marks** | |

## Question 8

10 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $\cos\theta = \dfrac{\lvert 2-1+1\rvert}{\sqrt{3}\sqrt{6}}$ | M1 |
| | $= \dfrac{2}{\sqrt{18}}$ | A1 |
| | $61.9^\circ$ | A1 |
| | | (3) |
| (b) | Direction $\mathbf{n}_1 \times \mathbf{n}_2 = 2\mathbf{i}+\mathbf{j}-3\mathbf{k}$ | M1 |
| | Correct | A1 |
| | Common point with $z=0$: $x+y=6$, $2x-y=3$, so $(3,3,0)$ | M1 |
| | $\mathbf{r} = 3\mathbf{i}+3\mathbf{j}+t(2\mathbf{i}+\mathbf{j}-3\mathbf{k})$ | A1 |
| | | (4) |
| (c) | $\dfrac{x-3}{2} = \dfrac{y-3}{1} = \dfrac{z}{-3}$ | B1 |
| | | (1) |
| (d) | $\mathbf{a}\times\mathbf{b} = (3\mathbf{i}+3\mathbf{j})\times(2\mathbf{i}+\mathbf{j}-3\mathbf{k})$ | M1 |
| | $\mathbf{r}\times(2\mathbf{i}+\mathbf{j}-3\mathbf{k}) = -9\mathbf{i}+9\mathbf{j}-3\mathbf{k}$ | A1 |
| | | (2) |
| | **Total for Question 8: 10 marks** | |

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## Question 9

10 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $\overrightarrow{AB} \times \overrightarrow{AC} = (2,-1,1) \times (1,2,1)$<br>$= -3\mathbf{i} - \mathbf{j} + 5\mathbf{k}$<br>$\boxed{\frac{1}{2}\sqrt{35}}$ | M1<br>A1<br>A1<br>(3) |
| (b) | $\overrightarrow{AD} = \mathbf{j} + 4\mathbf{k}$; triple product $-1 + 20 = 19$<br>$\frac{1}{6} \times 19$<br>$\boxed{\frac{19}{6}}$ | M1<br>M1<br>A1<br>(3) |
| (c) | $V = \frac{1}{3} \times \text{area} \times h$: $h = \dfrac{3 \times \frac{19}{6}}{\frac{1}{2}\sqrt{35}}$<br>$\boxed{\dfrac{19}{\sqrt{35}}}$ | M1<br>A1<br>(2) |
| (d) | $\boxed{3x + y - 5z = 4}$<br>$\dfrac{\lvert 3 + 2 - 20 - 4 \rvert}{\sqrt{35}} = \dfrac{19}{\sqrt{35}}$ * | B1<br>B1<br>(2) |
| | | **Total for Question 9: 10 marks** |

## Question 10

10 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $\mathbf{n} = (\mathbf{i}+\mathbf{j}) \times (\mathbf{j}+\mathbf{k}) = \mathbf{i}-\mathbf{j}+\mathbf{k}$<br>$\mathbf{n} \cdot \mathbf{i} = 1$<br>$\boxed{x-y+z=1}$ | M1<br>M1<br>A1<br>(3) |
| (b) | $(\mathbf{i}-\mathbf{k}) \cdot (\mathbf{i}-\mathbf{j}+\mathbf{k}) = 0$, and $(2,3,4)$ is not on $\Pi$<br>$\frac{\lvert 2-3+4-1 \rvert}{\sqrt{3}}$<br>$\boxed{\frac{2}{\sqrt{3}}}$ | M1<br>M1<br>A1<br>(3) |
| (c) | $t\mathbf{n}$ with $3t = 1$<br>$\boxed{\left(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}\right)}$ | M1<br>A1<br>(2) |
| (d) | Normal $(\mathbf{i}-\mathbf{k}) \times (\mathbf{i}-\mathbf{j}+\mathbf{k}) = -\mathbf{i}-2\mathbf{j}-\mathbf{k}$<br>$\boxed{x+2y+z=12}$ | M1<br>A1<br>(2) |
| | | **Total for Question 10: 10 marks** |

# 6 Chapter 6: Further Matrix Algebra

Mark scheme. Marks are shown on the right of each line.

## Question 1

6 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $2(6-0)-1(2k-1)+0$ | M1 |
| | $\boxed{13-2k}$ | A1 |
| | $13-2k=0$ | M1 |
| | $\boxed{k=\frac{13}{2}}$ | A1 |
| | | (4) |
| (b) | $\det\mathbf{M}=3;\ \det\left(\mathbf{M}^2\right)=(\det\mathbf{M})^2$ | M1 |
| | $\boxed{9}$ | A1 |
| | | (2) |
| | | **Total for Question 1: 6 marks** |

## Question 2

6 marks

| Part | Scheme | Marks |
|---|---|---|
| | $\det\mathbf{A}=1(1)-2(0-6)=13$ | B1 |
| | Matrix of minors (at least six correct) | M1 |
| | Cofactors $\begin{pmatrix}1&6&-2\\-2&1&4\\6&-3&1\end{pmatrix}$ | A1 |
| | Transposes (adjugate) | M1 |
| | Divides by the determinant | M1 |
| | $\boxed{\frac{1}{13}\begin{pmatrix}1&-2&6\\6&1&-3\\-2&4&1\end{pmatrix}}$ | A1 |
| | | (6) |
| | | **Total for Question 2: 6 marks** |

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## Question 3

6 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $(\mathbf{AB})\left(\mathbf{B}^{-1}\mathbf{A}^{-1}\right) = \mathbf{A}\left(\mathbf{BB}^{-1}\right)\mathbf{A}^{-1} = \mathbf{AA}^{-1}$ | M1 |
| | $= \mathbf{I}$, so $\mathbf{B}^{-1}\mathbf{A}^{-1}$ is the inverse of $\mathbf{AB}$ * | A1* |
| | | (2) |
| (b) | $\mathbf{A}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 6 & -3 & 1 \end{pmatrix}$ | B1 |
| | $\mathbf{B}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$ | B1 |
| | $\mathbf{B}^{-1}\mathbf{A}^{-1}$ in this order | M1 |
| | $\begin{pmatrix} 3 & -1 & 0 \\ -2 & 1 & 0 \\ 3 & -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$ | A1 |
| | | (4) |
| | | **Total for Question 3: 6 marks** |

## Question 4

6 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $\det(\mathbf{M} - \lambda\mathbf{I})$, expanding down the third column | M1 |
| | $(1-\lambda)\left[(4-\lambda)(3-\lambda)-2\right] = (1-\lambda)\left(\lambda^2 - 7\lambda + 10\right)$ | A1 |
| | $= (1-\lambda)(\lambda-2)(\lambda-5)$: eigenvalues 1, 2, 5 * | A1* |
| | | (3) |
| (b) | $(\mathbf{M} - 5\mathbf{I})\mathbf{x} = \mathbf{0}$: $-x + 2y = 0$, $x + y - 4z = 0$ | M1 |
| | $x = 2y$, $3y = 4z$ | A1 |
| | $\begin{pmatrix} 8 \\ 4 \\ 3 \end{pmatrix}$ | A1 |
| | | (3) |
| | | **Total for Question 4: 6 marks** |

## Question 5

6 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $\mathbf{M}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}3+k\\4\\4\end{pmatrix}$ | M1 |
| | $= \lambda\begin{pmatrix}1\\1\\1\end{pmatrix}$, so $\lambda = 4$ | A1 |
| | $3+k=4$ | M1 |
| | $\boxed{\lambda = 4,\ k = 1}$ | A1 |
| | | (4) |
| (b) | $1(6-0)-1(0-2)+2(0-6)$ | M1 |
| | $\boxed{-4}$ | A1 |
| | | (2) |
| | | **Total for Question 5: 6 marks** |

## Question 6

10 marks

| Part | Scheme | Marks |
|---|---|---|
| **(a)** | $\det(\mathbf{M}-4\mathbf{I}) = \det\begin{pmatrix}-2 & 0 & k\\ 0 & -1 & 0\\ k & 0 & -2\end{pmatrix} = 0$ | **M1** |
| | $k^2 - 4 = 0$ | **A1** |
| | $\boxed{k=-2}$ (since $k<0$) | **A1** |
| | *Jan 2022 report: substituting the given eigenvalue first gives the quadratic in $k$ most easily; use the stated sign of $k$.* | **(3)** |
| **(b)** | $(3-\lambda)\left[(2-\lambda)^2-4\right]=0$ | **M1** |
| | $(2-\lambda)^2 = 4$ | **A1** |
| | $\boxed{\lambda=0 \text{ and } \lambda=3}$ | **A1** |
| | | **(3)** |
| **(c)** | $(\mathbf{M}-4\mathbf{I})\mathbf{x}=\mathbf{0}$: $y=0$, $z=-x$ | **M1** |
| | $\boxed{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}}$ | **A1** |
| | *Jan 2022 report: divide by the magnitude to normalise; some multiplied by it.* | **(2)** |
| **(d)** | 0 is an eigenvalue | **B1** |
| | and $\det\mathbf{M}$ is the product of the eigenvalues, so $\det\mathbf{M}=0$ | **B1** |
| | | **(2)** |
| | | **Total for Question 6: 10 marks** |

## Question 7

10 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $\mathbf{M} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & 2 & -1 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$ | B1 |
| | $\det \mathbf{M} = 7$ | B1 |
| | Cofactors | M1 |
| | Adjugate and division | M1 |
| | $\frac{1}{7}\begin{pmatrix} -1 & 3 & 2 \\ 3 & -2 & 1 \\ 5 & -1 & -3 \end{pmatrix}$ | A1 |
| | | (5) |
| (b) | $\mathbf{x} = \mathbf{M}^{-1}\mathbf{c} = \frac{1}{7}\begin{pmatrix} -6+9+4 \\ 18-6+2 \\ 30-3-6 \end{pmatrix}$ | M1 |
| | Correct | A1 |
| | $(1, 2, 3)$ | A1 |
| | | (3) |
| (c) | $\mathbf{M}$ is non-singular, so there is a unique solution | B1 |
| | the three planes meet at exactly one point | B1 |
| | | (2) |
| | | **Total for Question 7: 10 marks** |

## Question 8

10 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $\det \mathbf{T} = 2(1) - 0 + 1(1) = 3$ | B1 |
| | Cofactors | M1 |
| | Correct | A1 |
| | Adjugate divided by 3 | M1 |
| | $\frac{1}{3}\begin{pmatrix} 1 & 1 & -1 \\ -1 & 2 & 1 \\ 1 & -2 & 2 \end{pmatrix}$ | A1 |
| | | (5) |
| (b) | $l: \mathbf{r} = \mathbf{i} - \mathbf{k} + t(2\mathbf{i} + \mathbf{j} + \mathbf{k})$ | B1 |
| | $\mathbf{T}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \mathbf{T}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$ | M1 |
| | $\mathbf{r} = \mathbf{i} + \mathbf{j} - \mathbf{k} + t(5\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$ | A1 |
| | | (3) |
| (c) | $\mathbf{T}^{-1}\begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$ | M1 |
| | $(1, 1, 2)$ | A1 |
| | | (2) |
| | | **Total for Question 8: 10 marks** |

## Question 9

10 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $(5-\lambda)\left[(3-\lambda)^2-1\right]=0$<br>$3-\lambda=\pm 1$<br>$\boxed{\lambda=2,4,5}$ | M1<br>A1<br>A1<br>(3) |
| (b) | $\lambda=2$: $x+y=0$, $z=0$; $\lambda=4$: $x=y$, $z=0$; $\lambda=5$: $x=y=0$<br>Two correct<br>$\boxed{\frac{1}{\sqrt{2}}(1,-1,0),\ \frac{1}{\sqrt{2}}(1,1,0),\ (0,0,1)}$ | M1<br>A1<br>A1<br>(3) |
| (c) | Columns are the normalised eigenvectors<br>$\boxed{\mathbf{P}=\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 1\end{pmatrix}}$<br>$\boxed{\mathbf{D}=\begin{pmatrix}2 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 5\end{pmatrix}}$ (same order) | M1<br>A1<br>A1<br>(3) |
| (d) | The columns of $\mathbf{P}$ are orthonormal, so $\mathbf{P}^{\mathrm{T}}\mathbf{P}=\mathbf{I}$ | B1<br>(1) |
| | | **Total for Question 9: 10 marks** |

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## Question 10

10 marks

| Part | Scheme | Marks |
|---|---|---|
| (a) | $\mathbf{A}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}4\\4\\4\end{pmatrix}$ | M1 |
| | eigenvalue 4 | A1 |
| | | (2) |
| (b) | $\det(\mathbf{A}-\lambda\mathbf{I}) = -\lambda^3+6\lambda^2-9\lambda+4$ | M1 |
| | Divides by $(\lambda-4)$: $-(\lambda-4)(\lambda-1)^2$ | M1 |
| | $\lambda = 1$ repeated * | A1* |
| | | (3) |
| (c) | $\lambda = 1$: $x+y+z=0$ (a plane of eigenvectors) | M1 |
| | Perpendicular pair, e.g. $(1,-1,0)$ and $(1,1,-2)$ (scalar product 0) | A1 |
| | $\mathbf{P} = \begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}}\end{pmatrix}$ | A1 |
| | $\mathbf{D} = \begin{pmatrix}4&0&0\\0&1&0\\0&0&1\end{pmatrix}$ | A1 |
| | | (4) |
| (d) | $4\times1\times1 = \boxed{4}$ | B1 |
| | | (1) |
| | | **Total for Question 10: 10 marks** |