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A-LEVEL MATHEMATICS • FURTHER PURE 3

Further Matrix Algebra

Worked Solutions

Pearson Edexcel International A Level (WFM03) • Calculator allowed: show every step

Full worked solutions for the free Further Matrix Algebra worksheet, A-Level and International A-Level Maths

A — **Examples** & Your turn — page 2

B — **Fluency** core skills — page 2

C — **Reasoning** explain it — page 4

D — **Problem** solving — page 5

E — **Exam** practice — page 6

Tap a section to jump straight to it

Full worked solutions for every question on the Further Pure 3 Further Matrix Algebra worksheet, in the same order and with the same question numbers.

How to mark your work. Tick each step you got right, not just the final answer. At A-level most marks are for method, so a wrong final answer can still earn most of the marks. If you lost marks, find the *first* line where your working is different from ours. That is the thing to fix.

In Section E: **M** = method mark, **dM** = method mark that depends on the previous M mark, **A** = accuracy mark (needs the M mark before it), **B** = independent mark.

Question sheet | Exam questions by topic | Further Pure 3 exam questions | More worksheets

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A Your turn answers

Compare each line of your working with the solution.

CORE

A1 $1(0-4)-0+2(12-2)$: $\boxed{16}$

A2 $\det = 1(2-3)-k(4-3) = -1-k$: $\boxed{k=-1}$

A3 $\boxed{\begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}}$

STRETCH

A4 $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$: $\boxed{\lambda = 2}$

A5 Triangular: $\boxed{1,\ -2,\ 5}$

A6 $(\mathbf{M}+\mathbf{I})\mathbf{x} = \mathbf{0}$: $2x+2z=0$, $4y=0$: $\boxed{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}$

A7 Length $\sqrt{6}$: $\boxed{\frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}}$

B Fluency solutions

One mark for each correct part. The notes in italics explain why each sequence was chosen.

CORE

B1 **(a)** $\boxed{2}$ **(b)** $\boxed{4}$ **(c)** $\boxed{1}$ **(d)** $2^3 \times 4$: $\boxed{32}$

Each row multiplied by 2 multiplies the determinant by 2.

B2 **(a)** $\boxed{5}$ **(b)** $\boxed{\begin{pmatrix} 1 & -2 & 2 \\ 2 & 1 & -1 \\ -2 & 4 & 1 \end{pmatrix}}$ **(c)** $\boxed{\begin{pmatrix} 1 & 2 & -2 \\ -2 & 1 & 4 \\ 2 & -1 & 1 \end{pmatrix}}$ **(d)** $\boxed{\frac{1}{5}\begin{pmatrix} 1 & 2 & -2 \\ -2 & 1 & 4 \\ 2 & -1 & 1 \end{pmatrix}}$

Cofactors, transpose, divide.

B3 **(a)** $\boxed{-6}$ **(b)** $\boxed{\frac{1}{3}}$ **(c)** $\boxed{-6}$ **(d)** $\boxed{24}$

Determinants multiply; transposing does not change them.

B4 **(a)** $k^2 - k + 6 = \left(k - \frac{1}{2}\right)^2 + \frac{23}{4} > 0$: **none** **(b)** $-(k-1)(k-2)$: $\boxed{k = 1, 2}$

(c) $k\left(k^2 - 2\right)$: $\boxed{k = 0, \pm\sqrt{2}}$ **(d)** volumes are mapped to zero, so there is no inverse transformation

Singular means det = 0

B5 **(a)** $\boxed{5}$ **(b)** $\boxed{-1}$ **(c)** $\boxed{-1}$: two independent eigenvectors share the repeated eigenvalue -1

Multiply and compare with the vector.

B6 **(a)** $\boxed{(2-\lambda)\left[(4-\lambda)^2 - 1\right]}$ **(b)** $\boxed{(2-\lambda)(3-\lambda)(5-\lambda)}$ **(c)** **2, 3, 5** **(d)** $\boxed{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}$

Expand along the row with zeros.

STRETCH

B7 **(a)** row 3: $1 + 1 + 3$: $\boxed{\lambda = 5}$ **(b)** $\boxed{a = 2, b = 2}$ **(c)** $\boxed{-5}$

(d) the others multiply to -1 and add to $6 - 5 = 1$: $\lambda^2 - \lambda - 1 = 0$: $\boxed{\frac{1 \pm \sqrt{5}}{2}}$

The product of the eigenvalues is $\det \mathbf{M}$*; their sum is the trace.*

B8 **(a)** $\mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}$ **(b)** $\frac{1}{5}\begin{pmatrix} 1 & 2 & -2 \\ -2 & 1 & 4 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}$: $\boxed{\left(\frac{7}{5}, \frac{6}{5}, \frac{9}{5}\right)}$ **(c)** substitute back **(d)** three planes meeting at one point

$\det \mathbf{A} \neq 0$ *gives exactly one solution.*

B9 **(a)** $\boxed{(4,4,1)}$ **(b)** $\boxed{\mathbf{r} = \begin{pmatrix}1\\0\\1\end{pmatrix} + t\begin{pmatrix}1\\2\\0\end{pmatrix}}$ **(c)** $\boxed{(1,2,2)}$ **(d)** $\boxed{-2}$: volumes are doubled and orientation reversed

The image of $\mathbf{a}+t\mathbf{b}$ *is* $\mathbf{Ma}+t\mathbf{Mb}$

B10 **(a)** $\boxed{\mathbf{I}}$ **(b)** $\boxed{\mathbf{P}^{-1}=\mathbf{P}^{\mathrm{T}}}$ **(c)** $\boxed{-1}$ **(d)** the entries of $\mathbf{P}^{\mathrm{T}}\mathbf{P}$ are the dot products of the columns: 1 on the diagonal, 0 elsewhere

An orthogonal matrix is inverted by transposing.

C Reasoning solutions

Your wording can be different. Check that your reason makes the same point.

CORE

C1 Eigenvalues satisfy $\det(\mathbf{M}-\lambda\mathbf{I})=0$, not $\det\mathbf{M}=0$

C2 The cofactor matrix must be transposed (the adjugate) before dividing by $\det\mathbf{A}$

C3 The order reverses: $(\mathbf{AB})\left(\mathbf{B}^{-1}\mathbf{A}^{-1}\right)=\mathbf{AIA}^{-1}=\mathbf{I}$, so $\boxed{(\mathbf{AB})^{-1}=\mathbf{B}^{-1}\mathbf{A}^{-1}}$

C4 Divide by the length 3, do not multiply: $\boxed{\frac{1}{3}\begin{pmatrix}2\\1\\2\end{pmatrix}}$

STRETCH

C5 It should be $(\mathbf{M}+\mathbf{I})\mathbf{x}=\mathbf{0}$ (equivalently $\mathbf{Mx}=-\mathbf{x}$): $\boxed{\begin{pmatrix}1\\0\\-1\end{pmatrix}}$

C6 The condition $k<0$ must be applied: $\boxed{k=-3}$

C7 They are eigenvectors, but not perpendicular (dot product 1), so $\mathbf{P}$ would not be orthogonal: $\boxed{\text{use } \begin{pmatrix}1\\-1\\0\end{pmatrix} \text{ and } \begin{pmatrix}1\\1\\-2\end{pmatrix}}$

C8 **(a)** $\mathbf{M}^2\mathbf{x} = \mathbf{M}(\lambda\mathbf{x}) = \lambda\mathbf{M}\mathbf{x} = \lambda^2\mathbf{x}$, as required

(b) $\lambda_1\mathbf{x}_1 \cdot \mathbf{x}_2 = (\mathbf{M}\mathbf{x}_1) \cdot \mathbf{x}_2 = \mathbf{x}_1 \cdot \left(\mathbf{M}^{\mathrm{T}}\mathbf{x}_2\right) = \mathbf{x}_1 \cdot (\mathbf{M}\mathbf{x}_2) = \lambda_2\mathbf{x}_1 \cdot \mathbf{x}_2$; since $\lambda_1 \neq \lambda_2$, $\mathbf{x}_1 \cdot \mathbf{x}_2 = 0$, as required

D Problem solving solutions

There is often more than one correct method. Any complete, correct method scores full marks.

CORE

D1 $1(2k-1) - k(0-3) + 2(0-6) = 5k - 13$: $\boxed{k = \frac{13}{5}}$

D2 **(a)** $\det\mathbf{A} = x^2 - x + 2 = \left(x - \frac{1}{2}\right)^2 + \frac{7}{4} > 0$, as required

(b) $\boxed{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & -1 \\ -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \end{pmatrix}}$

D3 $x = u$, $y = v - u$, $z = w - 2v + 2u$; substitute into $x + y + z = 4$: $\boxed{2u - v + w = 4}$

D4 **(a)** $(2-7)(5-7) - k^2 = 0$: $\boxed{k = \sqrt{10}}$

(b) 1 from the third row; the block has trace 7, so the other is 0: **0 and 1**

STRETCH

D5 **(a)** row 3: $\lambda = 3$; $1 + a + 3 = 3$, $2 + b = 3$: $\boxed{\lambda = 3,\ a = -1,\ b = 1}$

(b) $\det(\mathbf{M} - \lambda\mathbf{I}) = -(\lambda - 3)\left(\lambda^2 + 2\right)$, and $\lambda^2 = -2$ has no real roots, as required

D6 **(a)** $(2-\lambda)\left[(4-\lambda)(1-\lambda) + 2\right] = (2-\lambda)(\lambda-2)(\lambda-3)$: **2 (repeated) and 3**

(b) $\lambda = 3$: $\boxed{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}$; $\lambda = 2$: $x - 2y - z = 0$ gives two independent eigenvectors, e.g. $\boxed{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}$

D7 Row 1 of $\mathbf{T}$ with column 2 of $\mathbf{U}$: $2a + 6 = 0$; row 3 with column 1: $3 + 3b = 0$; row 3 with column 3: $1 + 3c = 7$: $\boxed{a = -3,\ b = -1,\ c = 2}$; $\mathbf{U} = \mathbf{T}^{-1}$

D8 **(a)** $\det(\mathbf{M}-\lambda\mathbf{I}) = (4-\lambda)\left[(k-\lambda)^2-1\right]$, giving the result

(b) $k+1=4$: $\boxed{k=3\text{, eigenvalues 2, 4, 4}}$; $k-1=4$: $\boxed{k=5\text{, eigenvalues 4, 4, 6}}$ ($k+1 \neq k-1$ always)

E Exam practice solutions

Mark-scheme notes are shown on the right of each line.

CORE *Exam questions for everyone*

E1 **(a)** $2(3-0)-1(1-2)+k(0-3) = 7-3k$ *M1*

$\boxed{k=\frac{7}{3}}$ *A1*

(b) cofactors, then transpose *M1 M1*

adjugate $\begin{pmatrix} 3 & -1 & 2-3k \\ 1 & 2-k & k-4 \\ -3 & 1 & 5 \end{pmatrix}$ *A1*

$\boxed{\mathbf{A}^{-1} = \frac{1}{7-3k}\begin{pmatrix} 3 & -1 & 2-3k \\ 1 & 2-k & k-4 \\ -3 & 1 & 5 \end{pmatrix}}$ *A1*

E2 **(a)** $\det(\mathbf{M}-\lambda\mathbf{I}) = (3-\lambda)\left[(2-\lambda)(3-\lambda)-1\right]+1\times\left[-(3-\lambda)\right]$ *M1 A1*

$= (3-\lambda)\left(\lambda^2-5\lambda+4\right)$ *M1*

$= (3-\lambda)(\lambda-1)(\lambda-4)$, as required *A1**

(b) $(\mathbf{M}-\mathbf{I})\mathbf{x}=\mathbf{0}$: $2x-y=0$, $-y+2z=0$ *M1*

$\boxed{\frac{1}{\sqrt{6}}\begin{pmatrix}1\\2\\1\end{pmatrix}}$ *A1*

$\begin{pmatrix}1\\2\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\1\end{pmatrix} = 0$ *B1*

STRETCH

E3 **(a)** $\mathbf{M}\begin{pmatrix}2\\2\\-1\end{pmatrix} = \begin{pmatrix}6\\6\\-3\end{pmatrix}$: $\boxed{\lambda = 3}$ *B1*

(b) trace 18 and $\det\mathbf{M} = 162$: the others add to 15 and multiply to 54 *M1 A1*

$\boxed{\textbf{6 and 9}}$ *A1*

(c) $\lambda = 6$: $\begin{pmatrix}2\\-1\\2\end{pmatrix}$; $\lambda = 9$: $\begin{pmatrix}-1\\2\\2\end{pmatrix}$ *M1 A1*

$\boxed{\frac{1}{3}\begin{pmatrix}2\\2\\-1\end{pmatrix}, \frac{1}{3}\begin{pmatrix}2\\-1\\2\end{pmatrix}, \frac{1}{3}\begin{pmatrix}-1\\2\\2\end{pmatrix}}$ *A1*

(d) $\boxed{\mathbf{P} = \frac{1}{3}\begin{pmatrix}2&2&-1\\2&-1&2\\-1&2&2\end{pmatrix}, \mathbf{D} = \begin{pmatrix}3&0&0\\0&6&0\\0&0&9\end{pmatrix}}$ *M1 A1*

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