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A-LEVEL MATHEMATICS • FURTHER PURE 1

Transformations Using Matrices

Worked Solutions

Pearson Edexcel International A Level (WFM01) • Calculator allowed: show every step

Full worked solutions for the free Transformations Using Matrices worksheet, A-Level and International A-Level Maths

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B — **Fluency** core skills — page 2

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Tap a section to jump straight to it

Full worked solutions for every question on the Further Pure 1 Transformations Using Matrices worksheet, in the same order and with the same question numbers.

How to mark your work. Tick each step you got right, not just the final answer. At A-level most marks are for method, so a wrong final answer can still earn most of the marks. If you lost marks, find the *first* line where your working is different from ours. That is the thing to fix.

In Section E: **M** = method mark, **dM** = method mark that depends on the previous M mark, **A** = accuracy mark (needs the M mark before it), **B** = independent mark.

Question sheet | Exam questions by topic | Further Pure 1 exam questions | More worksheets

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A Your turn answers

Compare each line of your working with the solution.

CORE

A1 Columns $\binom{0}{2}$ and $\binom{5}{-1}$: $\boxed{\begin{pmatrix} 0 & 5 \\ 2 & -1 \end{pmatrix}}$

A2 $\boxed{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \textbf{ and } \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}}$

A3 $\det = 4 + 2 = 6$; 6×7: $\boxed{\textbf{42}}$

STRETCH

A4 $(1,0) \to (-3,0)$, $(0,1) \to (0,-3)$: $\boxed{\textbf{enlargement, scale factor } -3\textbf{, centre } O}$

A5 $2\mathbf{I} \times \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$: $\boxed{\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}}$

A6 $\cos 120^\circ = -\frac{1}{2}$, $\sin 120^\circ = \frac{\sqrt{3}}{2}$: $\boxed{\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}}$

A7 $x + 2y = x$ and $-y = y$, so $y = 0$: $\boxed{\textbf{every point on the } x\textbf{-axis}}$

B Fluency solutions

One mark for each correct part. The notes in italics explain why each sequence was chosen.

CORE

B1 **(a)** $\boxed{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$ **(b)** $\boxed{\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}}$ **(c)** $\boxed{\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}}$ **(d)** $\boxed{\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}}$

Each column is where a unit vector goes.

B2 **(a)** $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ **(b)** $\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ **(c)** $\begin{pmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}$ **(d)** $\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

Clockwise is a negative angle; 330° *anticlockwise is* 30° *clockwise.*

B3 **(a)** **reflection in the y-axis** **(b)** **reflection in $y = -x$**

(c) $\cos\theta = -\frac{1}{2}$, $\sin\theta = -\frac{\sqrt{3}}{2}$: **rotation 120° clockwise about O**

(d) **stretch, scale factor 5, parallel to the x-axis**

A rotation matrix has equal diagonal entries; a reflection has opposite ones.

B4 **(a)** $(0,0)$, $(6,-2)$, $(1,2)$ **(b)** $\frac{1}{2}|6\times 2-(-2)(1)|$: 7 **(c)** $\det\mathbf{M} = 7$, **the same** **(d)** 28

The determinant is the area scale factor.

B5 **(a)** $\mathbf{QP}$: $\begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$ **(b)** $\mathbf{PQ}$: $\begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$ **(c)** both have determinant 2, as $\det(\mathbf{QP}) = \det\mathbf{Q}\det\mathbf{P}$

The order changes the matrix, not the area scale factor.

B6 **(a)** $\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ **(b)** $\mathbf{M}^{-1}\binom{5}{3}$: $(2,1)$ **(c)** $(x,x)\to(3x,2x)$: $y = \frac{2}{3}x$ **(d)** 1

The inverse undoes the transformation.

STRETCH

B7 **(a)** $a+2b=4$, $3a+b=7$: $a=2$, $b=1$ **(b)** $c+2d=7$, $3c+d=6$: $c=1$, $d=3$ **(c)** 5 **(d)** 5×4: 20

Each row of $\mathbf{M}$ *comes from one coordinate of the images.*

B8 **(a)** $\begin{pmatrix} 4-3 \\ 2-1 \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix}1$, as required **(b)** $4x-3y=x$ and $2x-y=y$: $y=x$ **(c)** $(4,2)$ **(d)** $-4+6$: 2

Invariant points satisfy $\mathbf{M}\binom{x}{y} = \binom{x}{y}$

B9 **(a)** $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ **(b)** $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ **(c)** **reflection in the x-axis** **(d)** **reflection in the y-axis**

Swapping the order gives a different reflection.

B10 **(a)** $\boxed{k^2-6}$ **(b)** $k^2-6=10$ (-10 has no real root): $\boxed{k=\pm 4}$ **(c)** $k^2=6$: $\boxed{k=\pm\sqrt{6}}$ **(d)** 10×3: $\boxed{30}$

A zero determinant collapses the plane onto a line.

C Reasoning solutions

Your wording can be different. Check that your reason makes the same point.

CORE

C1 The first transformation acts first on the vector, so it goes on the right: $\boxed{\mathbf{P}\text{ followed by }\mathbf{Q}\text{ is }\mathbf{QP}}$

C2 Areas are multiplied by the determinant, not divided: 12×3: $\boxed{36}$

C3 $(1,0)\to(0,1)$ and $(0,1)\to(-1,0)$; its determinant is $+1$, so it is not a reflection: $\boxed{\text{rotation } 90^\circ \text{ anticlockwise about } O}$

C4 The scale factor is negative and the centre must be given: $\boxed{\text{enlargement, scale factor } -2\text{, centre } O}$

STRETCH

C5 An area cannot be negative: use $|\det|$: $\boxed{12}$; the negative sign means the orientation is reversed (a reflection is involved)

C6 $\sin\theta=-\frac{1}{2}$, so the angle is negative: $\boxed{\text{rotation } 30^\circ \text{ clockwise about } O}$

C7 $\mathbf{A}$ is on the right of $\mathbf{C}$, so $\mathbf{C}=\begin{pmatrix}2&3\\1&2\end{pmatrix}\mathbf{A}^{-1}=\begin{pmatrix}2&3\\1&2\end{pmatrix}\begin{pmatrix}1&-1\\0&1\end{pmatrix}$: $\boxed{\begin{pmatrix}2&1\\1&1\end{pmatrix}}$

C8 **(a)** $\mathbf{R}(\alpha)\mathbf{R}(\beta)=\begin{pmatrix}\cos\alpha\cos\beta-\sin\alpha\sin\beta & -(\sin\alpha\cos\beta+\cos\alpha\sin\beta)\\ \sin\alpha\cos\beta+\cos\alpha\sin\beta & \cos\alpha\cos\beta-\sin\alpha\sin\beta\end{pmatrix}=\mathbf{R}(\alpha+\beta)$, as required

(b) $\mathbf{R}(\theta)\mathbf{R}(-\theta)=\mathbf{R}(0)=\mathbf{I}$, so $\mathbf{R}(\theta)^{-1}=\mathbf{R}(-\theta)$; $\det\mathbf{R}(\theta)=\cos^2\theta+\sin^2\theta=1$, so areas are unchanged

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D Problem solving solutions

There is often more than one correct method. Any complete, correct method scores full marks.

CORE

D1 **(a)** **enlargement, scale factor -4, centre O** **(b)** $\mathbf{Q} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(c) $\mathbf{PQ}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = -4\begin{pmatrix} 3 \\ 2 \end{pmatrix}$: $A(-12, -8)$

D2 **(a)** $\mathbf{M}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3+2k \\ k+4 \end{pmatrix} = \begin{pmatrix} 11 \\ 8 \end{pmatrix}$: $k = 4$

(b) $\det\mathbf{M} = 6 - 16 = -10$, area scale factor 10; $\frac{60}{10}$: **6 square units**

D3 **(a)** **enlargement: $p = q = 3$; reflection: $p = 1$, $q = -1$**

(b) $2p = -6$, $5q = 10$: $p = -3$, $q = 2$

D4 **(a)** $\mathbf{C} = \mathbf{BA} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$: $\begin{pmatrix} -\frac{\sqrt{3}}{2} & -1 \\ \frac{1}{2} & -\sqrt{3} \end{pmatrix}$

(b) $\det\mathbf{C} = \frac{3}{2} + \frac{1}{2} = 2$: **6 square units**

STRETCH

D5 **(a)** $\mathbf{W}\begin{pmatrix} \lambda \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda+2 \\ 3\lambda+2 \end{pmatrix} = \begin{pmatrix} 4\lambda \\ 4 \end{pmatrix}$: $\lambda + 2 = 4\lambda$ and $3\lambda + 2 = 4$ both give $\lambda = \frac{2}{3}$

(b) The line through O and $\left(\frac{2}{3}, 1\right)$: $y = \frac{3}{2}x$, with distances multiplied by 4

D6 **(a)** $\mathbf{A}^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$: rotation 90° anticlockwise about O

(b) $\mathbf{A}$ is a rotation of 45°, and $8 \times 45° = 360°$: $n = 8$ **(c)** $\mathbf{A}^4 = -\mathbf{I}$: **rotation 180° about O**

D7 $\det\mathbf{A} = k^2 - 10 = \pm 4$; $k^2 = 14$ or $k^2 = 6$: $k = \pm\sqrt{14}$ **or** $k = \pm\sqrt{6}$

D8 **(a)** $\mathbf{N} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

(b) $\mathbf{B} = \mathbf{NM}$: $\begin{pmatrix} 2 & 9 \\ 0 & 3 \end{pmatrix}$

(c) $\det \mathbf{B} = 6$; $\frac{72}{6}$: **12 square units**

E Exam practice solutions

Mark-scheme notes are shown on the right of each line.

CORE *Exam questions for everyone*

E1 **(a)** A reflection *M1*

in the line $y = -x$ *A1*

(b) $\mathbf{Q} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ *B1*

(c) $\mathbf{QP}\begin{pmatrix} 2 \\ 5 \end{pmatrix} = 3\begin{pmatrix} -5 \\ -2 \end{pmatrix}$ *M1*

$B(-15, -6)$ *A1*

(d) $\det(\mathbf{QP}) = -9$: **area scale factor 9**; the negative sign shows a reflection is involved *B1*

E2 **(a)** A rotation, 30° *M1*

anticlockwise about O *A1*

(b) $\mathbf{B} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ *B1*

(c) $\mathbf{C} = \mathbf{BA}$ *M1*

$\begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ *A1*

(d) W' is represented by $(2\mathbf{C})^{-1} = \frac{1}{2}\mathbf{C}^{-1}$; $\det \mathbf{C} = -1$, $\mathbf{C}^{-1} = \mathbf{C}$ *M1*

$\begin{pmatrix} -\frac{\sqrt{3}}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{\sqrt{3}}{4} \end{pmatrix}$ *A1*

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STRETCH

E3 **(a)** $\det\mathbf{M} = 4+2 = 6$; 6×5: $\boxed{30}$ *B1*

(b) $\mathbf{M}\begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} (4-m)x \\ (2+m)x \end{pmatrix}$ *M1*

on $y = mx$: $(2+m)x = m(4-m)x$ *M1*

$m^2 - 3m + 2 = 0$, as required *A1**

$(m-1)(m-2) = 0$: $\boxed{y = x \text{ and } y = 2x}$ *A1*

(c) $\mathbf{M}\begin{pmatrix} x \\ x \end{pmatrix} = \begin{pmatrix} 3x \\ 3x \end{pmatrix} = 3\begin{pmatrix} x \\ x \end{pmatrix}$, so every point on $y = x$ moves three times as far from O *B1*

(d) $\mathbf{N} = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}\mathbf{M}^{-1}$, with $\mathbf{M}^{-1} = \frac{1}{6}\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$ *M1 M1*

$\boxed{\mathbf{N} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}}$ *A1*