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A-LEVEL MATHEMATICS • FURTHER PURE 2

Complex Numbers

Worksheet

Pearson Edexcel International A Level (WFM02) • Calculator allowed: show every step

Free Complex Numbers worksheet for A-Level and International A-Level Maths, with full worked solutions

A Examples & Your turn page 2

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Tap a section to jump straight to it

By the end of this worksheet you can: use exponential form • apply de Moivre's theorem • find nth roots • *Stretch:* multiple-angle identities, powers of $\cos\theta$ and $\sin\theta$ in multiple angles, and equations solved with them.

Which questions are for me?

CORE The methods every A-level student must secure. Start here.

STRETCH Exponential form, de Moivre's theorem for identities and roots, and equations solved without a calculator.

Every section has both, clearly labelled. Questions get harder as you go, and every skill in Sections A to D is needed in Section E.

Track your progress

Section	B	C	D	E
Core marks	23	10	13	13
Stretch marks	16	14	17	9
My score				

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A Worked examples & Your turn

Read each example carefully. Cover it up, then answer the question beside it using the same method.

CORE *Secure the method*

Worked example A1: exponential form

Express $-2+2\mathrm{i}$ in the form $re^{\theta \mathrm{i}}$, $-\pi < \theta \leqslant \pi$

$r = \sqrt{8}$; second quadrant, so $\theta = \pi - \frac{\pi}{4}$

$2\sqrt{2}e^{\frac{3\pi}{4}\mathrm{i}}$

Your turn A1

Express $-\sqrt{3}-\mathrm{i}$ in the form $re^{\theta \mathrm{i}}$, $-\pi < \theta \leqslant \pi$

Worked example A2: multiply and divide

Simplify $\dfrac{4e^{\frac{\pi}{3}\mathrm{i}} \times 3e^{\frac{\pi}{4}\mathrm{i}}}{2e^{-\frac{\pi}{6}\mathrm{i}}}$

Multiply and divide the moduli; add and subtract the arguments

$6e^{(\frac{\pi}{3}+\frac{\pi}{4}+\frac{\pi}{6})\mathrm{i}}$

$6e^{\frac{3\pi}{4}\mathrm{i}}$

Your turn A2

Simplify $\dfrac{6e^{\frac{2\pi}{3}\mathrm{i}}}{3e^{-\frac{\pi}{4}\mathrm{i}}}$

Worked example A3: de Moivre's theorem

Find $(1-\mathrm{i})^6$

$\left(\sqrt{2}e^{-\frac{\pi}{4}\mathrm{i}}\right)^6 = 8e^{-\frac{3\pi}{2}\mathrm{i}}$

$8\mathrm{i}$

Your turn A3

Find $(-1+\sqrt{3}\mathrm{i})^5$ in the form $a+b\mathrm{i}$

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STRETCH

Worked example A4: a multiple angle

Show that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$

Real part of $(c+\mathrm{i}s)^4$*:* $c^4 - 6c^2s^2 + s^4$

$s^2 = 1 - c^2$

$8c^4 - 8c^2 + 1$

Your turn A4

Show that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

Worked example A5: powers in multiple angles

Express $\sin^3\theta$ in terms of $\sin 3\theta$ and $\sin\theta$

$\left(z - \frac{1}{z}\right)^3 = z^3 - z^{-3} - 3\left(z - z^{-1}\right)$ *with* $z - \frac{1}{z} = 2\mathrm{i}\sin\theta$

$-8\mathrm{i}\sin^3\theta = 2\mathrm{i}\sin 3\theta - 6\mathrm{i}\sin\theta$

$\frac{1}{4}(3\sin\theta - \sin 3\theta)$

Your turn A5

Express $\cos^5\theta$ in terms of $\cos 5\theta$, $\cos 3\theta$ and $\cos\theta$

Worked example A6: nth roots

Solve $z^4 = -16$

$16\mathrm{e}^{(\pi+2k\pi)\mathrm{i}}$*, so* $z = 2\mathrm{e}^{\left(\frac{\pi}{4}+\frac{k\pi}{2}\right)\mathrm{i}}$

$\pm\sqrt{2} \pm \sqrt{2}\mathrm{i}$

Your turn A6

Solve $z^3 = 27\mathrm{i}$, giving answers in the form $r\mathrm{e}^{\theta\mathrm{i}}$

Worked example A7: roots and geometry

Explain why the five roots of $z^5 = 1$ add up to 0

They are equally spaced around the unit circle, and $z^5 - 1$ *has no* z^4 *term*

sum of roots = 0

Your turn A7

Find the area of the square formed by the roots of $z^4 = -16$

B Fluency

Each question is a designed sequence. From one part to the next, ask yourself: what changed, and what did it do to the answer?

CORE *Secure the method*

B1 Express in the form $re^{\theta i}$, $-\pi < \theta \leqslant \pi$

(a) $3i$

(b) -5

(c) $1-\sqrt{3}i$

(d) $-3-3i$

[4 marks]

B2 $z = 2e^{\frac{\pi}{6}i}$ and $w = 4e^{-\frac{\pi}{4}i}$. Find, in exponential form,

(a) zw

(b) $\dfrac{z}{w}$

(c) z^6

(d) w^3

[4 marks]

B3 Use de Moivre's theorem to find

(a) $(1+i)^8$

(b) $(\sqrt{3}-i)^6$

(c) $(1+\sqrt{3}i)^{-3}$

(d) the least n with $(1+i)^n$ real and positive

[4 marks]

B4 Use de Moivre's theorem to express

(a) $\sin 4\theta$ in terms of $\sin\theta$ and $\cos\theta$

(b) $\cos 4\theta$ in powers of $\sin\theta$

(c) $\dfrac{\sin 5\theta}{\sin\theta}$ in powers of $\cos\theta$

(d) $\dfrac{\sin 6\theta}{\sin 2\theta}$ in terms of $\sin 2\theta$

[4 marks]

B5 $z = \cos\theta + i\sin\theta$. Show that

(a) $z^n + \dfrac{1}{z^n} = 2\cos n\theta$

(b) $z^n - \dfrac{1}{z^n} = 2i\sin n\theta$

(c) $\sin^2\theta\cos^2\theta = \dfrac{1}{8}(1-\cos 4\theta)$

[3 marks]

B6 Solve, giving roots in exponential form,

(a) $z^3 = 8$

(b) $z^4 = 81$

(c) $z^6 = -1$

(d) $z^2 = 2i$ (in the form $a+bi$)

[4 marks]

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STRETCH

B7 $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$

(a) Solve $8x^4 - 8x^2 + 1 = 0$, writing each root as $\cos\alpha$

(b) Find $\cos^2\frac{\pi}{8}$ exactly.

(c) Find the product of the four roots.

(d) Find $\cos\frac{\pi}{8}\cos\frac{3\pi}{8}$ exactly.

[4 marks]

B8 Use $z \pm \frac{1}{z}$ to

(a) express $\sin^4\theta$ in multiple angles

(b) find $\int_0^{\frac{\pi}{2}} \sin^4\theta \, d\theta$

(c) express $\cos^3\theta$ in multiple angles

(d) find $\int_0^{\frac{\pi}{3}} \cos^3\theta \, d\theta$

[4 marks]

B9 $z^6 = 64$

(a) Find the six roots.

(b) Find their sum.

(c) Find the area of the hexagon they form.

(d) Find their product.

[4 marks]

B10 $z^3 = -4\sqrt{2} + 4\sqrt{2}\,\mathrm{i}$

(a) Write $-4\sqrt{2} + 4\sqrt{2}\,\mathrm{i}$ in exponential form.

(b) Find the three roots.

(c) Write one root in the form $a + b\mathrm{i}$

(d) Check it by cubing.

[4 marks]

C Reasoning

Show your working and explain your thinking. A correct answer without a reason does not score full marks.

CORE *Secure the method*

C1 Amir writes $\arg(-1-\mathrm{i}) = \dfrac{\pi}{4}$. Explain his error, and give the argument. **[2 marks]**

C2 Bea writes $\left(2\mathrm{e}^{\frac{\pi}{3}\mathrm{i}}\right)^4 = 8\mathrm{e}^{\frac{4\pi}{3}\mathrm{i}}$. Explain her error, and give the answer with $-\pi < \theta \leqslant \pi$ **[2 marks]**

C3 Cal writes the real part of $(\cos\theta + \mathrm{i}\sin\theta)^3$ as $\cos^3\theta - \sin^2\theta\cos\theta$

Explain his error, and give the correct expression for $\cos 3\theta$ in powers of $\cos\theta$ **[3 marks]**

C4 Dev solves $z^4 = -81$ and gives the answer $z = 3\mathrm{e}^{\frac{\pi}{4}\mathrm{i}}$

Explain why his answer is incomplete, and give all the roots. **[3 marks]**

STRETCH

C5 Solving $4\sin^2\theta = 1$ for $0 < \theta < \dfrac{\pi}{2}$, Eve writes $\sin\theta = \dfrac{1}{4}$

Explain her error, and solve the equation. **[3 marks]**

C6 Asked for the smallest positive root of $8x^4 - 8x^2 + 1 = 0$, Fay gives $\cos\dfrac{\pi}{8}$

Explain her error, and give the correct root to 3 decimal places. **[3 marks]**

C7 Gus gives one root of an equation as $2\mathrm{e}^{\frac{7\pi}{6}\mathrm{i}}$

Explain why this is not in the required form $-\pi < \theta \leqslant \pi$, and correct it. **[3 marks]**

C8 **(a)** Prove by induction that $(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta$ for all positive integers n

(3)

(b) Show that the result also holds for negative integers n

(2)

[5 marks]

D Problem solving

You decide the method. Read the whole question first, and show every step.

CORE *Secure the method*

D1 The complex number z is given by $z = -3\sqrt{3} + 3\mathrm{i}$

(a) Express z in the form $re^{\theta \mathrm{i}}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$, giving the exact values of r and θ

(2)

(b) Hence, using de Moivre's theorem and making your method clear, determine z^4 in the form $a + b\mathrm{i}$, where a and b are exact real numbers.

(1)

[3 marks]

D2 *In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.*

Solve the equation

$$z^4 + 8 + 8\sqrt{3}\mathrm{i} = 0$$

giving your answers in the form $re^{\theta \mathrm{i}}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$ **[4 marks]**

D3 The complex number z is given by $z = -7 + 24\mathrm{i}$

(a) Determine the modulus of z, and show that the argument of z lies between $\frac{\pi}{2}$ and π

(1)

(b) Determine the values of w such that $w^2 = z$, giving your answers in the form $a + \mathrm{i}b$, where a and b are real numbers.

(2)

[3 marks]

D4 *In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.*

Without using your calculator, and showing each stage of your working in modulus-argument or exponential form, show that

$$\frac{\left(\cos\frac{\pi}{9} + \mathrm{i}\sin\frac{\pi}{9}\right)^6}{\left(\cos\frac{\pi}{6} - \mathrm{i}\sin\frac{\pi}{6}\right)^2} = -1$$

[3 marks]

STRETCH

D5 **(a)** Use de Moivre's theorem, by expanding $(\cos\theta + \mathrm{i}\sin\theta)^4$ and comparing real and imaginary parts, to show that $\tan 4\theta \equiv \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$

(2)

(b) Hence, by first showing that this equation is equivalent to $\tan 4\theta = 1$ where $t = \tan\theta$, determine the solutions of $t^4 + 4t^3 - 6t^2 - 4t + 1 = 0$, giving your answers to 3 decimal places.

(2)

[4 marks]

D6 *In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.*

(a) Using $z - \dfrac{1}{z} = 2\mathrm{i}\sin\theta$, where $z = \cos\theta + \mathrm{i}\sin\theta$, show that $16\sin^5\theta \equiv \sin 5\theta - 5\sin 3\theta + 10\sin\theta$

(2)

(b) Hence determine the exact value of $\int_0^{\frac{\pi}{2}} \sin^5\theta \, \mathrm{d}\theta$

(2)

[4 marks]

D7 *In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.*

Given that $\cos 6\theta \equiv 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$,

(a) show that, with $u = \cos^2\theta$, the equation $\cos 6\theta = 0$ becomes $(2u - 1)(16u^2 - 16u + 1) = 0$

(1)

(b) hence determine the exact values of $\cos^2\dfrac{\pi}{12}$ and $\cos^2\dfrac{5\pi}{12}$

(3)

[4 marks]

D8 **(a)** Express $4\sqrt{3} + 4\mathrm{i}$ in the form $r\mathrm{e}^{\theta\mathrm{i}}$

(1)

(b) Hence solve $z^3 = 4\sqrt{3} + 4\mathrm{i}$, giving your answers in the form $r\mathrm{e}^{\theta\mathrm{i}}$, $-\pi < \theta \leqslant \pi$

(3)

(c) The points representing the three roots on an Argand diagram are the vertices of a triangle. Determine the exact area of this triangle.

(1)

[5 marks]

E Exam practice

Exam-style questions in the style of WFM02. Write your answers in the spaces, and show every step.

CORE *Exam questions for everyone*

E1 *In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.*

(a) Express $-16\sqrt{3}-16\mathrm{i}$ in the form $r\mathrm{e}^{\theta\mathrm{i}}$, where $r>0$ and $-\pi<\theta\leqslant\pi$

(2)

(b) Hence solve

$$z^5=-16\sqrt{3}-16\mathrm{i}$$

giving your answers in the form $p\mathrm{e}^{\theta\mathrm{i}}$, where $p\in\mathbb{R}^+$ and $-\pi<\theta\leqslant\pi$

(4)

(Total for Question E1 is 6 marks)

E2 The complex numbers z and w are given by $z=1+\mathrm{i}$ and $w=\sqrt{3}-\mathrm{i}$

(a) Express z and w in the form $r\mathrm{e}^{\theta\mathrm{i}}$, where $r>0$ and $-\pi<\theta\leqslant\pi$

(2)

(b) Hence express $\dfrac{z}{w}$ in the form $r\mathrm{e}^{\theta\mathrm{i}}$

(1)

Using de Moivre's theorem, and making your method clear,

(c) determine, in simplest form, the value of z^8w^6

(2)

(d) determine the smallest positive integer n for which $\left(\dfrac{z}{w}\right)^n$ is real.

(2)

(Total for Question E2 is 7 marks)

STRETCH

E3 *In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.*

(a) Use de Moivre's theorem to show that

$$\cos 3\theta \equiv \cos^3\theta - 3\cos\theta\sin^2\theta \quad \text{and} \quad \sin 3\theta \equiv 3\cos^2\theta\sin\theta - \sin^3\theta$$

(3)

(b) Hence show that, for $\cos 3\theta \neq 0$,

$$\tan 3\theta \equiv \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

(2)

(c) Using the result of part (b), determine the exact solutions of the equation

$$t^3 - 3t^2 - 3t + 1 = 0$$

(4)

(Total for Question E3 is 9 marks)

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