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INTERNATIONAL A LEVEL MATHEMATICS • FURTHER PURE 3 (WFM03)

Further Pure 3 Practice Paper 3

Mark Scheme

AIMING FOR A/A*

Written in the style of the Pearson Edexcel International Advanced Level paper (WFM03/01) • 75 marks

Every question marked line by line the way an examiner marks it, with notes on the common errors

Mark every line, not just the answer. Method marks are given for a correct method even when the final answer is wrong. If you lost marks, find the *first* line where your working differs from the scheme. That is the thing to fix.

Code	Meaning
M1	Method mark: for knowing a correct method and attempting to apply it. Small slips in the working do not lose this mark.
dM1	Dependent method mark: can only be given if the method mark before it has been given.
A1	Accuracy mark: for a correct answer or line of working. It depends on the method mark before it.
B1	Independent mark: for a correct result or statement, whatever method was used.
*	The answer is given in the question. Every step must be shown, and the answer must follow from correct working (cso).
cao / cso	Correct answer only / correct solution only, with no errors anywhere in the working.
ft / oe	Follow through from your own earlier answer / or any correct equivalent form.

"Solutions relying on calculator technology are not acceptable." Questions 1 and 7 say this. There, a correct final answer with no algebraic working scores **no marks**. Elsewhere a calculator may be used, but the method must still be seen.

Rough grade guide (out of 75): A* **57** • A **51** • B **44** • C **37**

Based on recent WFM03 grade boundaries. Real boundaries change every series, so treat this as a guide, not a grade.

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What to revise next

Write in your score for each question. Where you dropped marks, tap the chapter to practise ten more exam questions on it.

Q	What it tested	Marks	Practise it	Score
1	Choosing $R\sinh(x+\alpha)$ over $R\cosh(x+\alpha)$; a rejected and a repeated root	10	Ch 1 Hyperbolic Functions	
2	The domain of an inverse hyperbolic function; integration by parts	10	Ch 3 Differentiation · Ch 4 Integration	
3	A repeated eigenvalue and orthogonal eigenvectors	11	Ch 6 Further Matrix Algebra	
4	A reflection in a plane; choosing the right parallel plane	11	Ch 5 Vectors	
5	A reduction formula whose boundary term does not vanish	11	Ch 4 Integration	
6	A tangent to an ellipse; least area; a restricted locus	11	Ch 2 Further Coordinate Systems	
7	Arc length and surface area with a hyperbolic substitution	11	Ch 4 Integration	
	Total	**75**		

Where did the marks go? At A* every question has one decision that costs marks if it is skipped: which form $R\sinh$ or $R\cosh$ fits, which root of a quadratic in e^x is impossible, where a function is defined, which pair of eigenvectors for a repeated eigenvalue is orthogonal, which of two parallel planes is meant, and which part of a curve a locus covers. Find the line where your working first differs from the scheme, then practise that chapter.

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Question 1

10 marks

Part	Scheme	Marks
(a)	$R\cosh(x+\alpha) = R\cosh\alpha\cosh x + R\sinh\alpha\sinh x$ needs $\tanh\alpha = \frac{5}{3} > 1$, which is impossible	**B1**
	$R\sinh(x+\alpha) = R\cosh\alpha\sinh x + R\sinh\alpha\cosh x$: $R\cosh\alpha = 5$, $R\sinh\alpha = 3$	**M1**
	$R^2 = 5^2 - 3^2 = 16$, $R = 4$	**A1**
	$\tanh\alpha = \frac{3}{5}$, $\alpha = \frac{1}{2}\ln\frac{1+\frac{3}{5}}{1-\frac{3}{5}} = \frac{1}{2}\ln 4$; $\boxed{f(x) = 4\sinh(x+\ln 2)}$	**A1**
		(4)
(b)	$\sinh(x+\ln 2) = 1$	**M1**
	$x + \ln 2 = \operatorname{arsinh} 1 = \ln\left(1+\sqrt{2}\right)$	**M1**
	$\boxed{x = \ln\frac{1+\sqrt{2}}{2}}$	**A1**
	Through exponentials: $4e^{2x} - 4e^x - 1 = 0$, $e^x = \frac{1\pm\sqrt{2}}{2}$. The negative value must be rejected, with a reason ($e^x > 0$), for the A1.	**(3)**
(c)	$5\cosh x + 3\sinh x = 4\cosh(x+\ln 2)$ (same R and α), so $\cosh(x+\ln 2) = 1$	**M1**
	$\boxed{x = -\ln 2}$	**A1**
	$\cosh u = 1$ only when $u = 0$, the least value of cosh; or $(2e^x - 1)^2 = 0$ is a repeated root	**B1**
		(3)
		Total for Question 1: 10 marks

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Question 2

10 marks

Part	Scheme	Marks
(a)	$-1 < \dfrac{x-1}{x+1} < 1$	M1
	$\boxed{x > 0}$	A1
		(2)
(b)	$f'(x) = \dfrac{1}{1-\left(\frac{x-1}{x+1}\right)^2} \times \dfrac{d}{dx}\left(\dfrac{x-1}{x+1}\right)$	M1
	$\dfrac{(x+1)^2}{(x+1)^2-(x-1)^2} \times \dfrac{2}{(x+1)^2} = \dfrac{2}{4x}$	A1
	$\boxed{f'(x) = \dfrac{1}{2x}}$	A1
	f(1) = artanh 0 = 0, so $\boxed{f(x) = \frac{1}{2}\ln x}$ for $x > 0$	B1
		(4)
(c)	$\left[\frac{1}{2}x^2 f(x)\right] - \int \frac{1}{2}x^2 \times \dfrac{1}{2x}\,dx$	M1 A1
	$\left[\frac{1}{2}x^2 f(x) - \frac{1}{8}x^2\right]_1^e$ with f(e) = $\frac{1}{2}$, f(1) = 0	M1
	$\frac{e^2}{4} - \frac{e^2}{8} + \frac{1}{8} = \boxed{\dfrac{e^2+1}{8}}$	A1
		(4)
		Total for Question 2: 10 marks

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Question 3

11 marks

Part	Scheme	Marks
(a)	$\mathbf{A}\begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}5\\-5\\5\end{pmatrix}$	M1
	$= 5\begin{pmatrix}1\\-1\\1\end{pmatrix}$, eigenvalue 5	A1
		(2)
(b)	$\det(\mathbf{A}-\lambda\mathbf{I})$ expanded	M1
	$-\lambda^3+9\lambda^2-24\lambda+20=0$	A1
	$-(\lambda-5)\left(\lambda^2-4\lambda+4\right)=0$	M1
	$-(\lambda-5)(\lambda-2)^2=0$: $\lambda=2$ (repeated)	A1
		(4)
(c)	$\lambda=2$: every row of $\mathbf{A}-2\mathbf{I}$ gives $x-y+z=0$	M1
	One eigenvector, e.g. (1, 1, 0)	A1
	A second that is *orthogonal* to it: $(1,1,0)\times(1,-1,1)$	M1
	$=(1,-1,-2)$	A1
	$\mathbf{P}=\begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}}\end{pmatrix}$, $\mathbf{D}=\begin{pmatrix}5&0&0\\0&2&0\\0&0&2\end{pmatrix}$	A1
	Any two orthogonal vectors in the plane $x-y+z=0$ are correct. Two eigenvectors for $\lambda=2$ that are not orthogonal, such as $(1,1,0)$ and $(1,0,-1)$, do not give an orthogonal $\mathbf{P}$: last A1 lost.	
		(5)
		Total for Question 3: 11 marks

Question 4

11 marks

Part	Scheme	Marks
(a)	$\mathbf{r} = (5, -1, 5) + t(2, -1, 2)$	M1
	$2(5+2t) - (-1-t) + 2(5+2t) = 3,\ \ 21+9t = 3$	M1
	$t = -2$: foot of the perpendicular $(1, 1, 1)$	A1
	A' at $t = -4$ (or $A' = 2(1,1,1) - A$)	M1
	$\boxed{A'(-3, 3, -3)}$	A1
		(5)
(b)	Π_2: $2x - y + 2z = d,\ \ \dfrac{\lvert d-3\rvert}{3} = 2$	M1
	$d = 9$ or $d = -3$	A1
	At A, $2x - y + 2z = 21$, so between A and Π_1 means $3 < d < 21$: $\boxed{2x - y + 2z = 9}$	A1
		(3)
(c)	At A': $2(-3) - 3 + 2(-3) = -15$	M1
	$\dfrac{\lvert -15-9\rvert}{3}$	M1
	$\boxed{8}$	A1
	Or: A' is 6 from Π_1 on the far side, and Π_2 is 2 from Π_1 on A's side, so $6+2=8$.	
		(3)
		Total for Question 4: 11 marks

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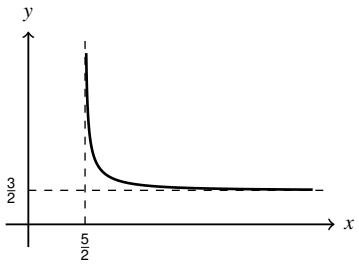
Question 5

11 marks

Part	Scheme	Marks
(a)	$\tanh^n x = \tanh^{n-2} x\left(1-\operatorname{sech}^2 x\right)$	**M1 A1**
	$I_n = I_{n-2} - \displaystyle\int_0^{\ln 3} \tanh^{n-2} x \operatorname{sech}^2 x \,\mathrm{d}x$	**M1**
	$= I_{n-2} - \left[\dfrac{\tanh^{n-1} x}{n-1}\right]_0^{\ln 3}$	**A1**
	$\tanh(\ln 3) = \dfrac{3-\frac{1}{3}}{3+\frac{1}{3}} = \dfrac{4}{5}$, $\tanh 0 = 0$: result	**A1***
		(5)
(b)	$I_1 = [\ln \cosh x]_0^{\ln 3}$	**M1**
	$\cosh(\ln 3) = \frac{5}{3}$: $\boxed{I_1 = \ln \frac{5}{3}}$	**A1**
		(2)
(c)	$I_3 = I_1 - \frac{1}{2}\left(\frac{4}{5}\right)^2 = \ln \frac{5}{3} - \frac{8}{25}$	**M1**
	$I_5 = I_3 - \frac{1}{4}\left(\frac{4}{5}\right)^4$	**M1**
	$\boxed{I_5 = \ln \frac{5}{3} - \frac{264}{625}}$	**A1**
		(3)
(d)	$0 \leqslant \tanh x < 1$ on $[0, \ln 3]$, so $\tanh^6 x \leqslant \tanh^5 x$, with equality only at $x = 0$	**B1**
		(1)
		Total for Question 5: 11 marks

Question 6

11 marks

Part	Scheme	Marks
(a)	$\frac{dy}{dx} = \frac{3\cos\theta}{-5\sin\theta}$	M1 A1
	$y - 3\sin\theta = -\frac{3\cos\theta}{5\sin\theta}(x - 5\cos\theta)$	M1
	$3x\cos\theta + 5y\sin\theta = 15$	A1
		(4)
(b)	$A\left(\frac{5}{\cos\theta}, 0\right)$, $B\left(0, \frac{3}{\sin\theta}\right)$	B1
	Area $= \frac{1}{2} \times \frac{5}{\cos\theta} \times \frac{3}{\sin\theta} = \frac{15}{\sin 2\theta}$	M1
	Least when $\sin 2\theta = 1$, $\theta = \frac{\pi}{4}$: least area 15	A1
	$P\left(\frac{5\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$	A1
		(4)
(c)	$M\left(\frac{5}{2\cos\theta}, \frac{3}{2\sin\theta}\right)$	M1
	$\cos^2\theta + \sin^2\theta = 1$: $\frac{25}{4x^2} + \frac{9}{4y^2} = 1$	M1
	$9x^2 + 25y^2 = 4x^2y^2$, with $x > \frac{5}{2}$, $y > \frac{3}{2}$	A1
		
	$0 < \theta < \frac{\pi}{2}$ gives $0 < \cos\theta < 1$ and $0 < \sin\theta < 1$. At $\theta = 0$ or $\frac{\pi}{2}$ the tangent is parallel to an axis and A or B does not exist.	
		(3)
	Total for Question 6: 11 marks	

Question 7

11 marks

Part	Scheme	Marks
(a)	$\frac{dy}{dx} = x,\ s = \int_0^{\sqrt{3}} \sqrt{1+x^2}\,dx$	M1
	$x = \sinh u$: $\int \cosh^2 u\,du$	M1
	Limits 0 and $\operatorname{arsinh}\sqrt{3} = \ln\left(2+\sqrt{3}\right)$, where $\cosh u = 2$	A1
	$\frac{1}{2}[u + \sinh u \cosh u]$	M1
	$\frac{1}{2}\left(\ln\left(2+\sqrt{3}\right) + 2\sqrt{3}\right) = \boxed{\sqrt{3} + \frac{1}{2}\ln\left(2+\sqrt{3}\right)}$	A1
		(5)
(b)	$S = 2\pi \int_0^{\sqrt{3}} \frac{1}{2}x^2\sqrt{1+x^2}\,dx$	M1
	$= \pi \int_0^{\ln(2+\sqrt{3})} \sinh^2 u \cosh^2 u\,du$	A1
	$\sinh^2 u \cosh^2 u = \frac{1}{4}\sinh^2 2u = \frac{1}{8}(\cosh 4u - 1)$	M1
	$\frac{\pi}{8}\left[\frac{1}{4}\sinh 4u - u\right]$	M1
	$\sinh 2u = 2\sqrt{3} \times 2 = 4\sqrt{3},\ \cosh 2u = 7,\ \sinh 4u = 56\sqrt{3}$	M1
	$\frac{\pi}{8}\left(14\sqrt{3} - \ln\left(2+\sqrt{3}\right)\right) = \boxed{\pi\left(\frac{7\sqrt{3}}{4} - \frac{1}{8}\ln\left(2+\sqrt{3}\right)\right)}$	A1
	Common losses: the 2π, the $\frac{1}{2}$ in y, and limits left in x after the substitution.	
		(6)
		Total for Question 7: 11 marks

TOTAL FOR PAPER: 75 MARKS

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