



eClassroom

A-LEVEL MATHEMATICS • PURE 4

Integration

Worksheet

Pearson Edexcel International A Level (WMA14) & A Level (9MA0) • Calculator allowed: show every step

Free Integration worksheet for A-Level and International A-Level Maths, with full worked solutions



Tap a section to jump straight to it

By the end of this worksheet you can: integrate by substitution, changing the limits • integrate by parts, with $\ln x$ as u • integrate using partial fractions • find volumes of revolution • solve separable differential equations • *Stretch:* parametric areas and volumes, parts twice, and modelling.

Which questions are for me?

CORE The methods every A-level student must secure. Start here.

STRETCH Parts twice, substitutions that lead to parts, parametric areas and volumes, and models you solve and interpret.

Every section has both, clearly labelled. Questions get harder as you go, and every skill in Sections A to D is needed in Section E.

Track your progress

Section	B	C	D	E
Core marks	23	10	13	11
Stretch marks	16	14	17	8
My score				

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A

Worked examples & Your turn

Read each example carefully. Cover it up, then answer the question beside it using the same method.

CORE

Secure the method

Worked example A1: substitution

Use $u = 2x - 1$ to find $\int_1^5 \frac{x}{\sqrt{2x-1}} dx$

$$x = \frac{u+1}{2}, dx = \frac{1}{2} du; \text{ the limits become } u = 1 \text{ and } u = 9$$

$$\frac{1}{4} \int_1^9 \left(u^{\frac{1}{2}} + u^{-\frac{1}{2}} \right) du = \frac{1}{4} \left[\frac{2}{3} u^{\frac{3}{2}} + 2u^{\frac{1}{2}} \right]_1^9$$

$$\frac{1}{4} \left(24 - \frac{8}{3} \right) = \frac{16}{3}$$

Your turn A1

Use $u = x + 1$ to find $\int_0^3 x\sqrt{x+1} dx$

Worked example A2: by parts

Find $\int_0^{\frac{\pi}{2}} x \sin x dx$

$$u = x, \frac{dv}{dx} = \sin x: \int x \sin x dx = -x \cos x + \int \cos x dx$$

$$[-x \cos x + \sin x]_0^{\frac{\pi}{2}} = (0 + 1) - (0 + 0)$$

$$1$$

Your turn A2

Find $\int_1^e x \ln x dx$, choosing u carefully.

Worked example A3: partial fractions

Find $\int_2^3 \frac{5}{(x-1)(x+4)} dx$, as a single logarithm.

$$\text{Split first: } \frac{5}{(x-1)(x+4)} = \frac{1}{x-1} - \frac{1}{x+4}$$

$$[\ln(x-1) - \ln(x+4)]_2^3 = (\ln 2 - \ln 7) - (0 - \ln 6)$$

$$\ln \frac{12}{7}$$

Your turn A3

Find $\int_0^1 \frac{3}{(x+1)(2-x)} dx$, as a single logarithm.



STRETCH

Worked example A4: volume of revolution

$y = x + \frac{1}{x}$ for $1 \leq x \leq 2$ is rotated 2π about the x -axis. Find the exact volume.

Square the whole of y : $y^2 = x^2 + 2 + \frac{1}{x^2}$

$$\pi \left[\frac{x^3}{3} + 2x - \frac{1}{x} \right]_1^2 = \pi \left(\frac{37}{6} - \frac{4}{3} \right)$$

$$\frac{29\pi}{6}$$

Your turn A4

$y = 1 + e^{-x}$ for $0 \leq x \leq \ln 2$ is rotated 2π about the x -axis. Find the exact volume.

Worked example A5: differential equations

Solve $\frac{dy}{dx} = xy^2$, given that $y = 1$ when $x = 0$

Separate, integrate both sides, add $+c$ at once.

$$\int y^{-2} dy = \int x dx: -\frac{1}{y} = \frac{x^2}{2} + c, \text{ and } c = -1$$

$$y = \frac{2}{2 - x^2}$$

Your turn A5

Solve $\frac{dy}{dx} = 2x(y + 1)$, given that $y = 1$ when $x = 0$

Worked example A6: parts twice

Find $\int_0^1 x^2 e^{2x} dx$

Parts twice: $u = x^2$, then $u = x$

$$\left[e^{2x} \left(\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4} \right) \right]_0^1 = \frac{e^2}{4} - \frac{1}{4}$$

$$\frac{e^2 - 1}{4}$$

Your turn A6

Find $\int_0^{\frac{\pi}{2}} x^2 \cos x dx$

Worked example A7: a parametric volume

$x = t^2$, $y = 2t$, $0 \leq t \leq 1$, is rotated 2π about the x -axis. Find the volume.

$V = \pi \int y^2 \frac{dx}{dt} dt$, with t limits

$$\pi \int_0^1 4t^2 \times 2t dt = 8\pi \left[\frac{t^4}{4} \right]_0^1$$

$$2\pi$$

Your turn A7

$x = 2 \sin t$, $y = \cos t$, $0 \leq t \leq \frac{\pi}{2}$, is rotated 2π about the x -axis. Find the volume.



B**Fluency**

Each question is a designed sequence. From one part to the next, ask yourself: what changed, and what did it do to the answer?

CORE*Secure the method*

B1 Use the given substitution to find each integral.

(a) $\int 2x(x^2 - 3)^5 dx$, $u = x^2 - 3$

(b) $\int \cos x e^{\sin x} dx$, $u = \sin x$

(c) $\int \frac{x}{x^2 + 4} dx$, $u = x^2 + 4$

(d) $\int x\sqrt{x-2} dx$, $u = x - 2$

[4 marks]

B2 Find the exact value of each integral, using a suitable substitution.

(a) $\int_0^1 x(1+x^2)^3 dx$

(b) $\int_0^{\frac{\pi}{2}} \sin x \cos^3 x dx$

(c) $\int_1^e \frac{\ln x}{x} dx$

(d) $\int_0^3 \frac{x}{\sqrt{x+1}} dx$

[4 marks]

B3 Use integration by parts to find

(a) $\int x e^{2x} dx$

(b) $\int x \cos 3x dx$

(c) $\int \ln x dx$

(d) $\int x^2 \ln x dx$

[4 marks]

B4 Use partial fractions to find

(a) $\int \frac{2}{(x-1)(x+1)} dx$

(b) $\int \frac{3}{x(x+3)} dx$

(c) $\int \frac{1}{(2x+1)(x+1)} dx$

(d) $\int \frac{x+2}{x+1} dx$

[4 marks]

B5 Each curve is rotated 2π about the x -axis between the given limits. Find each exact volume.

(a) $y = \sqrt{x}$, $0 \leq x \leq 4$

(b) $y = x^2$, $0 \leq x \leq 1$

(c) $y = \frac{1}{x}$, $1 \leq x \leq 3$

[3 marks]



B6 Solve each differential equation with the given condition, giving y (or y^2) in terms of x

(a) $\frac{dy}{dx} = 3y$, $y = 2$ when $x = 0$

(b) $\frac{dy}{dx} = \frac{x}{y}$, $y = 3$ when $x = 4$

(c) $\frac{dy}{dx} = y \cos x$, $y = e$ when $x = 0$

(d) $\frac{dy}{dx} = 2xe^{-y}$, $y = 0$ when $x = 0$

[4 marks]

STRETCH

B7 Find

(a) $\int \frac{e^{2x}}{e^x + 1} dx$, using $u = e^x + 1$

(b) $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + \sin x} dx$, exactly

(c) $\int \frac{1}{x + \sqrt{x}} dx$, using $u = \sqrt{x}$

(d) $\int_0^2 x^2 \sqrt{x^3 + 1} dx$, exactly

[4 marks]

B8 Find

(a) $\int x^2 e^{-x} dx$

(b) $\int x(x+1)^5 dx$

(c) $\int \frac{\ln x}{x^2} dx$

(d) $\int x \sin x \cos x dx$

[4 marks]

B9 Find the exact volume when each region is rotated 2π about the x -axis.

(a) Under $y = \sec x$, $0 \leq x \leq \frac{\pi}{3}$

(b) Under $y = e^x + e^{-x}$, $0 \leq x \leq 1$

(c) Under $y = \sin x$, $0 \leq x \leq \pi$

(d) Between $y = 1 + x^2$ and $y = 2$

[4 marks]

B10 Solve each differential equation with the given condition.

(a) $\frac{dy}{dx} = \frac{y-2}{x}$, $y = 5$ when $x = 1$

(b) $\cos y \frac{dy}{dx} = \sin x$, $y = 0$ when $x = \frac{\pi}{2}$

(c) $y \frac{dy}{dx} = xe^x$, $y = 1$ when $x = 0$

(d) $\frac{dy}{dx} = \frac{2y}{x(x+2)}$, $y = 1$ when $x = 2$

[4 marks]



C

Reasoning

Show your working and explain your thinking. A correct answer without a reason does not score full marks.

CORE

Secure the method

C1 To find $\int x(x^2 + 1)^3 dx$ with $u = x^2 + 1$, Ada writes $\int x u^3 du$

Explain her error, and find the integral.

[2 marks]

C2 Ben writes $\int \frac{4}{2x-1} dx = 4 \ln|2x-1| + c$

Explain his error, and give the correct answer.

[2 marks]

C3 For the volume when $y = 1 + \sqrt{x}$, $0 \leq x \leq 4$, is rotated about the x -axis, Cai writes $\pi \int_0^4 (1+x) dx$

Explain his error, and find the correct exact volume.

[3 marks]

C4 Dani solves $\frac{dy}{dx} = 2y$, with $y = 3$ when $x = 0$, and writes “ $\ln y = 2x$, so $y = e^{2x}$ ”

Explain why her answer cannot satisfy the condition, and give the correct solution.

[3 marks]

STRETCH

C5 To find $\int_0^2 x e^{x^2} dx$, Eve uses $u = x^2$ and gets $\frac{1}{2} [e^u]_0^2$

Find her error, and give the correct exact value.

[3 marks]

C6 To find $\int x \ln x dx$, Fin chooses $u = x$ and $\frac{dv}{dx} = \ln x$

Explain why this choice is poor, and find the integral with a better choice.

[3 marks]

C7 For $x = t^2$, $y = 2t$, $0 \leq t \leq 1$, Gita finds the volume of revolution as $\pi \int_0^1 4t^2 dt = \frac{4\pi}{3}$

Explain her error, and give the correct volume.

[3 marks]

C8 (a) Use integration by parts to show that $\int \ln x dx = x \ln x - x + c$

(2)

(b) Hence show that $\int_1^e (\ln x)^2 dx = e - 2$

(3)

[5 marks]



D

Problem solving

You decide the method. Read the whole question first, and show every step.

CORE

Secure the method

- D1** In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.

Use integration by parts to find the exact value of

$$\int_0^{\frac{\pi}{3}} x \sin 3x \, dx$$

giving your answer in its simplest form. You must state your choice of u and $\frac{dv}{dx}$, and show each stage of your integration clearly. [3 marks]

- D2** The function f is defined by $f(x) = \frac{5x+4}{(x-1)(x+2)}$, $x > 1$

(a) Express $f(x)$ in partial fractions, showing clearly how you find each constant.

(2)

(b) Using your answer to part (a) and algebraic integration, show that $\int_2^4 f(x) \, dx = \ln k$, where k is a rational number to be found. Give k as a fraction in its simplest form.

(2)

[4 marks]

- D3** A curve C passes through the point $(1, 3)$. At every point (x, y) on C the gradient satisfies the differential equation

$$\frac{dy}{dx} = \frac{6x^2}{y}, \quad y > 0$$

By solving the differential equation, find an equation for C , giving your answer in the form $y^2 = f(x)$, where $f(x)$ is a polynomial to be found. [3 marks]

- D4** Figure 1 shows the curve C with equation

$$y = \sqrt{x} + \frac{1}{\sqrt{x}}, \quad x > 0$$

The curve has a minimum point where $x = 1$. The region R is bounded by C , the x -axis and the lines $x = 1$ and $x = 4$. The region R is rotated through 2π radians about the x -axis. Use algebraic integration to find the exact volume of the solid generated, giving your answer in the form $\pi(a + b \ln 2)$, where a and b are rational constants to be found.

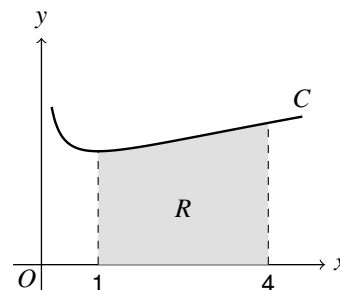


Figure 1

[3 marks]



STRETCH

- D5** Using the substitution $u = \sqrt{x}$, or otherwise, find the exact value of

$$\int_1^4 e^{\sqrt{x}} dx$$

giving your answer in its simplest form. You must show clearly how the limits and dx change. Solutions based entirely on graphical or numerical methods are not acceptable. [4 marks]

- D6** Figure 2 shows the curve C with parametric equations

$$x = t^2, \quad y = \sin t, \quad 0 \leq t \leq \pi$$

where t is a parameter. The region R is bounded by C and the x -axis.

- (a) Show that the area of the region R is given by the integral

$$\int_0^\pi 2t \sin t \, dt$$

(2)

- (b) Hence, using algebraic integration, find the exact area of R , giving your answer in terms of π

(2)

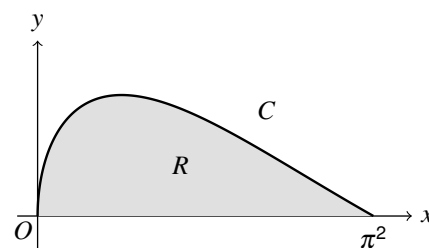


Figure 2

[4 marks]

- D7** A cup of tea at 90°C is left in a room at 20°C . Its temperature $\theta^\circ\text{C}$, t minutes later, satisfies

$$\frac{d\theta}{dt} = -k(\theta - 20)$$

where k is a positive constant. The model assumes that the temperature of the room stays constant. After 10 minutes the temperature of the tea is 55°C .

- (a) Show that $\theta = 20 + 70e^{-kt}$

(2)

- (b) Find the exact value of k , and use the model to find the time at which the temperature of the tea is 30°C , giving your answer to 3 significant figures.

(2)

[4 marks]

- D8** In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.

Figure 3 shows the curve C with equation $y = xe^{\frac{x}{2}}$. The region R is bounded by C , the x -axis and the line $x = 1$. The region R is rotated through 2π radians about the x -axis.

- (a) Show that the volume of the solid is given by

$$\pi \int_0^1 x^2 e^x dx$$

(1)

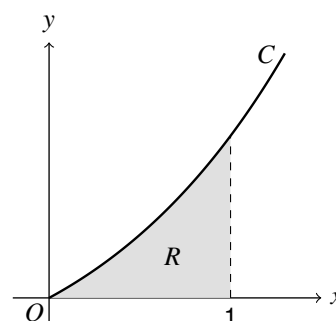


Figure 3

- (b) Using algebraic integration, find the exact volume of the solid, giving your answer in its simplest form.

(4)

[5 marks]



E**Exam practice**

Exam-style questions in the style of WMA14. Write your answers in the spaces, and show every step.

CORE

Exam questions for everyone

- E1** *In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.*

Using the substitution $u = 2x + 1$, and showing all stages of your working, show that

$$\int_0^4 \frac{x}{(2x+1)^2} dx = a \ln 3 + b$$

where a and b are rational constants to be found. You must show clearly how the limits change and how dx is replaced, and you should give a and b as fractions in their simplest form.

(5)

(Total for Question E1 is 5 marks)



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E2 Figure 4 shows part of the curve C with equation

$$y = \frac{3}{\sqrt{(x+1)(x+2)}}, \quad x \geq 0$$

The region R is bounded by C , the coordinate axes and the line $x = 2$

(a) Express $\frac{9}{(x+1)(x+2)}$ in partial fractions, showing clearly how you find each constant.

(2)

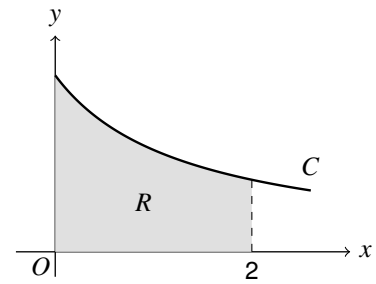


Figure 4

(b) The region R is rotated through 2π radians about the x -axis. Use your answer to part (a) and algebraic integration to find the exact volume of the solid generated, giving your answer in the form $p\pi \ln q$, where p and q are rational numbers.

(4)

(Total for Question E2 is 6 marks)



STRETCH

E3 In a chemical reaction, x grams of a product have formed after t minutes. The reaction is modelled by

$$\frac{dx}{dt} = k(3-x)(2-x), \quad 0 \leq x < 2$$

where k is a positive constant. Initially, when $t = 0$, no product has formed.

(a) Express $\frac{1}{(3-x)(2-x)}$ in partial fractions.

(2)

(b) Hence, using algebraic integration, show that $kt = \ln\left(\frac{2(3-x)}{3(2-x)}\right)$

(3)

(c) Given that $x = 1$ when $t = 5$, find the exact value of k

(1)

(d) Find x in terms of t , and hence state, with a reason, the mass of product that is formed in the long term, according to this model.

(2)

(Total for Question E3 is 8 marks)

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