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INTERNATIONAL A LEVEL MATHEMATICS • PURE 2 (WMA12)

Pure 2 Practice Paper 2

Worked Solutions

AIMING FOR A

Written in the style of the Pearson Edexcel International Advanced Level paper (WMA12/01) • 75 marks

Every question solved in full on the paper itself, the way your teacher would write it out

How to read these solutions. Each page is the real question page, with a full answer written on the answer lines.

Red pen

M1 A1

boxed

the working a full-marks answer shows, with a short note on why each step is taken

the mark earned on that line: M = method, A = accuracy, B = independent mark

the final answer to each part

Sit the paper first. These solutions are most useful after you have tried every question yourself under timed conditions. Mark your paper with the mark scheme, then use this booklet to see exactly how each lost mark could have been earned.

Q	Topic	Marks	Page
1	Binomial coefficients in an arithmetic sequence	8	2
2	Factor theorem, trigonometric equation	8	3
3	Stationary points and a circle	9	4
4	Logarithms of a geometric sequence	8	6
5	Exponential and trigonometric equations	8	8
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7	Trapezium rule, exponential curve	8	10
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Question paper & mark scheme | Exam questions by topic | More practice papers

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1. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

In the binomial expansion of

$$(1+kx)^7$$

where k is a non-zero constant, the coefficients of x, x^2 and x^3 are the first three terms of an arithmetic sequence.

(a) Find the possible values of k.

(5)

(b) Given that the arithmetic sequence is decreasing, find the sum of its first 20 terms.

(3)

(a) The general term is ${}^7C_r(kx)^r$, so the coefficients of x, x², x³ are:

${}^7C_1k = 7k$, ${}^7C_2k^2 = 21k^2$, ${}^7C_3k^3 = 35k^3$ B1

In an arithmetic sequence the middle term is the mean: $2(21k^2) = 7k + 35k^3$ M1

$35k^3 - 42k^2 + 7k = 0 \Rightarrow 7k(5k^2 - 6k + 1) = 0$ A1

$k \neq 0$, so divide by 7k: $(5k - 1)(k - 1) = 0$ dM1

$\boxed{k = \frac{1}{5} \text{ or } k = 1}$ A1

(b) $k = 1$ gives 7, 21, 35 (increasing). $k = \frac{1}{5}$ gives $\frac{7}{5}, \frac{21}{25}, \frac{7}{25}$: $a = \frac{7}{5}$, $d = \frac{21}{25} - \frac{35}{25} = -\frac{14}{25}$ B1

$S_{20} = \frac{20}{2}[2(\frac{7}{5}) + 19(-\frac{14}{25})] = 10[\frac{70}{25} - \frac{266}{25}]$ M1

$= 10 \times (-\frac{196}{25}) = \boxed{-\frac{392}{5}}$ $(= -78.4)$ A1

(Total for Question 1 is 8 marks)

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2. In this question you must show all stages of your working.
Solutions relying on calculator technology are not acceptable.

$$f(x) = 6x^3 - 7x^2 - 7x + 6$$

(a) Factorise $f(x)$ completely.

(3)

(b) Hence solve, for $0 \leqslant \theta < 360^\circ$,

$$6\cos^3\theta + 7\sin^2\theta = 7\cos\theta + 1$$

giving your answers to one decimal place where appropriate.

(5)

(a) Try small factors: $f(-1) = -6 - 7 + 7 + 6 = 0$, so $(x + 1)$ is a factor M1

Divide: $6x^3 - 7x^2 - 7x + 6 = (x + 1)(6x^2 - 13x + 6)$ M1

$6 \times 6 = 36$ and $-4 - 9 = -13$: $\boxed{f(x) = (x + 1)(2x - 3)(3x - 2)}$ A1

(b) Everything else is in $\cos\theta$, so replace $\sin^2\theta$ by $1 - \cos^2\theta$:

$6\cos^3\theta + 7(1 - \cos^2\theta) = 7\cos\theta + 1 \Rightarrow 6\cos^3\theta + 7 - 7\cos^2\theta - 7\cos\theta - 1 = 0$ M1

$6\cos^3\theta - 7\cos^2\theta - 7\cos\theta + 6 = 0$, which is $f(\cos\theta) = 0$ with $x = \cos\theta$ A1

So by (a): $(\cos\theta + 1)(2\cos\theta - 3)(3\cos\theta - 2) = 0 \Rightarrow \cos\theta = -1, \frac{3}{2}, \frac{2}{3}$ dM1

$\cos\theta = \frac{2}{3}$: $\theta = 48.19^\circ$, $360^\circ - 48.19^\circ = 311.81^\circ$ A1

$\cos\theta = -1$: $\theta = 180^\circ$. $\cos\theta = \frac{3}{2}$ is impossible ($\cos\theta \leqslant 1$), so reject it.

$\boxed{\theta = 48.2^\circ, 180^\circ, 311.8^\circ}$ A1

(Total for Question 2 is 8 marks)

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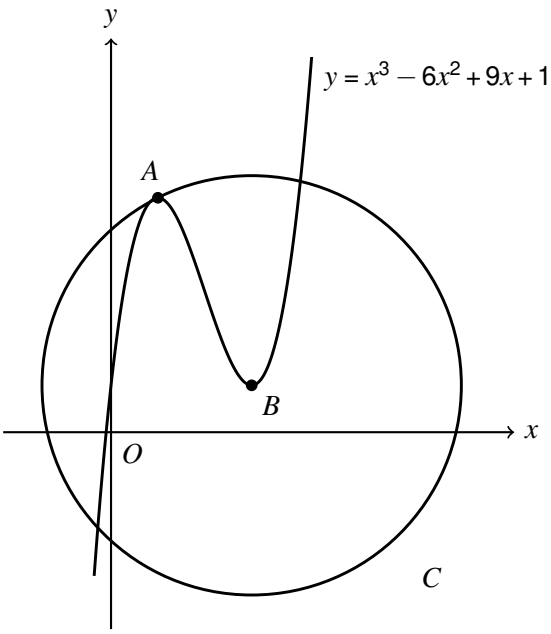

Figure 1

In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

Figure 1 shows a sketch of the curve with equation

$$y = x^3 - 6x^2 + 9x + 1$$

The curve has a maximum turning point at A and a minimum turning point at B.

(a) Use calculus to find the coordinates of A and the coordinates of B.

(3)

The circle C has centre B and passes through A, as shown in Figure 1.

(b) Find an equation for C.

(2)

The tangent to C at A meets the curve again at two points.

(c) Find the exact x coordinates of these two points.

(4)

(a) $\frac{dy}{dx} = 3x^2 - 12x + 9$ M1

$\frac{dy}{dx} = 0$: $3(x^2 - 4x + 3) = 0 \Rightarrow 3(x - 1)(x - 3) = 0 \Rightarrow x = 1$ or $x = 3$ M1

$x = 1$: $y = 1 - 6 + 9 + 1 = 5$; $x = 3$: $y = 27 - 54 + 27 + 1 = 1$

The maximum is on the left for a positive cubic: A(1, 5) and B(3, 1) A1

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Question 3 continued

(b) Radius^2 = BA^2 = $(3-1)^2 + (1-5)^2 = 4 + 16 = 20$ M1

$\boxed{(x-3)^2 + (y-1)^2 = 20}$ A1

(c) Radius BA has gradient $\frac{5-1}{1-3} = -2$, so the tangent at A (perpendicular) has gradient $\frac{1}{2}$ M1

Tangent: $y - 5 = \frac{1}{2}(x-1) \Rightarrow y = \frac{1}{2}x + \frac{9}{2}$ A1

Meets the curve: $x^3 - 6x^2 + 9x + 1 = \frac{1}{2}x + \frac{9}{2} \Rightarrow 2x^3 - 12x^2 + 17x - 7 = 0$

$x = 1$ (point A) is a root, so divide by $(x-1)$: $(x-1)(2x^2 - 10x + 7) = 0$ M1

Quadratic formula on $2x^2 - 10x + 7 = 0$: $x = (10 \pm \sqrt{44}) \div 4$

$\sqrt{44} = 2\sqrt{11}$, so $x = (10 \pm 2\sqrt{11}) \div 4$: $\boxed{x = \frac{1}{2}(5 \pm \sqrt{11})}$ A1

(Total for Question 3 is 9 marks)

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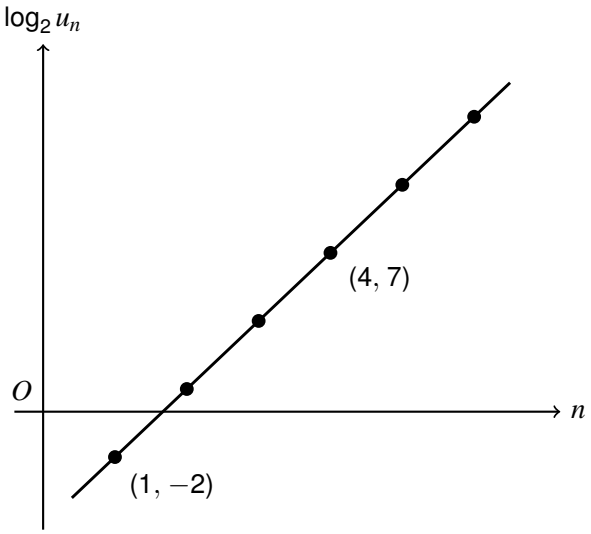

Figure 2

A geometric sequence $u_1, u_2, u_3, \ldots$ has positive terms.

Figure 2 shows the points with coordinates $(n, \log_2 u_n)$ for $n = 1, 2, \ldots, 6$

The points lie on a straight line, which passes through $(1, -2)$ and $(4, 7)$.

(a) Find the first term and the common ratio of the geometric sequence.

(3)

(b) Find the value of $\sum_{n=1}^{20} \log_2 u_n$

(2)

(c) Find the least value of n for which the sum of the first n terms of the geometric sequence exceeds 10^9

(3)

(a) $u_n = ar^{n-1}$, so $\log_2 u_n = \log_2 a + (n-1)\log_2 r$: a straight line with gradient $\log_2 r$

Gradient $= \frac{7-(-2)}{4-1} = 3$, so $\log_2 r = 3$ M1

$r = 2^3 = 8$ A1

At $n = 1$: $\log_2 u_1 = -2$, so the first term is $u_1 = 2^{-2} = \frac{1}{4}$ B1

(b) $\log_2 u_n = -2 + 3(n-1)$: an arithmetic sequence, first term -2, $d = 3$

$\sum \log_2 u_n$ for $n = 1$ to $20 = \frac{20}{2}[2(-2) + 19(3)] = 10 \times 53$ M1

$= 530$ A1

(c) $S_n = \frac{a(r^n - 1)}{r - 1} = \frac{8^n - 1}{4 \times 7} > 10^9 \Rightarrow 8^n - 1 > 2.8 \times 10^{10}$ M1

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Question 4 continued

$8^n > 2.8 \times 10^{10} + 1 \Rightarrow n \log 8 > \log(2.8 \times 10^{10} + 1) \Rightarrow n > 11.57...$ M1

$\boxed{n = 12}$ (check: $S_{11} = 3.07 \times 10^8$, $S_{12} = 2.45 \times 10^9$) A1

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(Total for Question 4 is 8 marks)

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5. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

(i) Solve, for $0 \leqslant x < 2\pi$,

$$9^{\sin^2 x} = 27^{\cos x}$$

giving your answers as exact multiples of π.

(4)

(ii) Solve, for $0 \leqslant \theta < 180^\circ$,

$$3\sin(2\theta + 40^\circ) = 2\cos(2\theta + 40^\circ)$$

giving your answers to one decimal place.

(4)

(i) Write both sides as powers of 3: $9 = 3^2$, $27 = 3^3$

$3^{2\sin^2 x} = 3^{3\cos x} \Rightarrow 2\sin^2 x = 3\cos x$ M1

$2(1 - \cos^2 x) = 3\cos x \Rightarrow 2\cos^2 x + 3\cos x - 2 = 0$ M1

$(2\cos x - 1)(\cos x + 2) = 0$: $\cos x = \frac{1}{2}$, ($\cos x = -2$ is impossible) A1

$\cos x = \frac{1}{2}$ in $0 \leqslant x < 2\pi$: $\boxed{x = \frac{\pi}{3}, \frac{5\pi}{3}}$ A1

(ii) Divide by $\cos(2\theta + 40^\circ)$: $\tan(2\theta + 40^\circ) = \frac{2}{3}$ M1

$0 \leqslant \theta < 180^\circ$ means $40^\circ \leqslant 2\theta + 40^\circ < 400^\circ$. $\tan^{-1}(\frac{2}{3}) = 33.69^\circ$ is too small, so add 180°:

$2\theta + 40^\circ = 213.69^\circ, 393.69^\circ$ M1

$2\theta = 173.69^\circ, 353.69^\circ \Rightarrow \theta = 86.85^\circ, 176.85^\circ$ dM1

$\boxed{\theta = 86.8^\circ, 176.8^\circ}$ A1

(Total for Question 5 is 8 marks)

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6. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

$$f(x) = x^3 + ax^2 + 3x$$

where a is a constant.

(a) Find the set of values of a for which $f(x)$ is an increasing function for all values of x.

(3)

Given now that $a = -4$

(b) sketch the curve with equation $y = f(x)$, showing the coordinates of the points where the curve meets the x-axis.

(2)

(c) Find the exact total area of the regions enclosed between the curve and the x-axis.

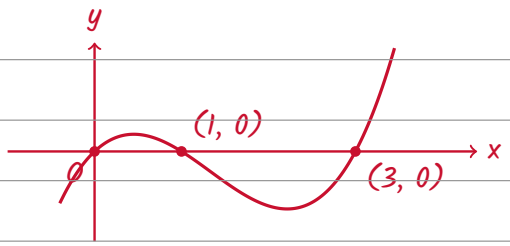
(4)

(a) $f'(x) = 3x^2 + 2ax + 3$. Increasing for all x needs $f'(x) \geqslant 0$ for all x (M1)

A positive quadratic that is never negative: $b^2 - 4ac \leqslant 0 \Rightarrow 4a^2 - 36 \leqslant 0$ (M1)

$a^2 \leqslant 9 \Rightarrow \boxed{-3 \leqslant a \leqslant 3}$ (A1)

(b) $f(x) = x^3 - 4x^2 + 3x = x(x^2 - 4x + 3) = x(x - 1)(x - 3)$: roots 0, 1, 3



(B1) (B1)

(c) $\int (x^3 - 4x^2 + 3x)\,dx = \frac{x^4}{4} - \frac{4x^3}{3} + \frac{3x^2}{2}$ (M1)

0 to 1: $\frac{1}{4} - \frac{4}{3} + \frac{3}{2} = \frac{5}{12}$. 1 to 3: $\left(\frac{81}{4} - 36 + \frac{27}{2}\right) - \frac{5}{12} = -\frac{9}{4} - \frac{5}{12} = -\frac{8}{3}$ (A1)

The second region is below the axis, so take its size: total $= \frac{5}{12} + \frac{8}{3}$ (M1)

$= \frac{5}{12} + \frac{32}{12} = \boxed{\frac{37}{12}}$ (not $-\frac{9}{4}$, which lets the regions cancel) (A1)

(Total for Question 6 is 9 marks)

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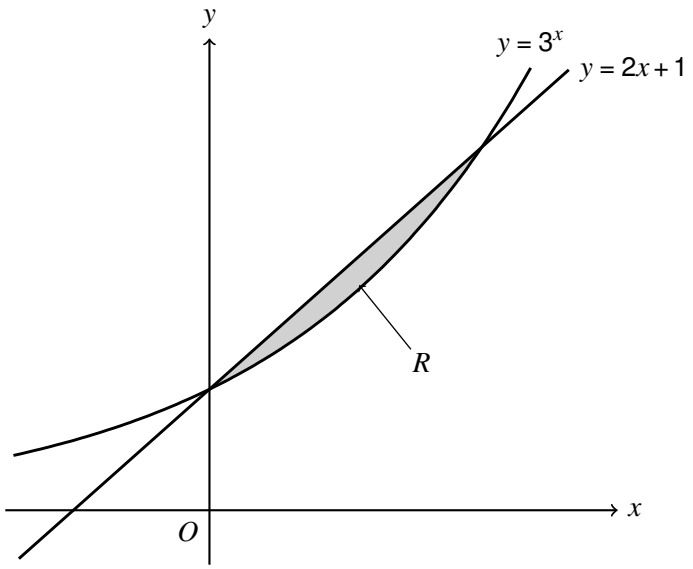

Figure 3

Figure 3 shows the curve with equation $y = 3^x$ and the line with equation $y = 2x + 1$

The region R, shown shaded in Figure 3, is bounded by the curve and the line.

(a) Show that the curve and the line meet where $x = 0$ and where $x = 1$

(1)

(b) Use the trapezium rule with four strips of equal width to find an estimate for

$$\int_0^1 3^x \, dx$$

giving your answer to 4 significant figures.

(3)

(c) Hence find an estimate for the area of R.

(2)

(d) State, with a reason, whether your answer to part (c) is an overestimate or an underestimate of the area of R.

(2)

(a) $x = 0$: $3^0 = 1 = 2(0) + 1$ ✓ $x = 1$: $3^1 = 3 = 2(1) + 1$ ✓ so they meet there **B1**

(b) 4 strips: $h = 0.25$. $y = 3^x$ at $x = 0, 0.25, 0.5, 0.75, 1$: $1, 1.31607, 1.73205, 2.27951, 3$ **B1**

$\int 3^x \, dx \approx \frac{1}{2} \times 0.25 \times [1 + 3 + 2(1.31607 + 1.73205 + 2.27951)]$ **M1**

$= 0.125 \times [4 + 10.65526] = \boxed{1.832}$ **A1**

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Question 7 continued

(c) Area under the line from 0 to 1: $\int (2x + 1)\,dx = [x^2 + x] = 2 - 0 = 2$ M1

R = (area under line) − (area under curve) $\approx 2 - 1.832 =$ 0.168 A1

(d) $y = 3^x$ bends upwards, so each trapezium lies above the curve: 1.832 is too big. B1

$R = 2 -$ (too big), so the answer to (c) is too small: an underestimate B1

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(Total for Question 7 is 8 marks)

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8. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

(a) Use the binomial expansion to write $\left(1+\sqrt{2}\right)^5$ in the form $a+b\sqrt{2}$, where a and b are integers.

(3)

(b) Prove that, for every positive integer n,

$$\left(1+\sqrt{2}\right)^n+\left(1-\sqrt{2}\right)^n$$

is an integer.

(3)

(c) Using your answer to part (a), explain why $\frac{41}{29}$ is an approximation to $\sqrt{2}$ that is correct to within 0.001

(2)

(a) $(1+\sqrt{2})^5 = 1 + 5\sqrt{2} + 10(\sqrt{2})^2 + 10(\sqrt{2})^3 + 5(\sqrt{2})^4 + (\sqrt{2})^5$ M1

$(\sqrt{2})^2 = 2$, $(\sqrt{2})^3 = 2\sqrt{2}$, $(\sqrt{2})^4 = 4$, $(\sqrt{2})^5 = 4\sqrt{2}$: $1 + 5\sqrt{2} + 20 + 20\sqrt{2} + 20 + 4\sqrt{2}$ M1

$41 + 29\sqrt{2}$ A1

(b) $(1+\sqrt{2})^n = \sum {}^nC_r(\sqrt{2})^r$ and $(1-\sqrt{2})^n = \sum {}^nC_r(-\sqrt{2})^r$ M1

r odd: $(-\sqrt{2})^r = -(\sqrt{2})^r$, so those terms cancel. r even: equal, so they double. M1

For $r = 2k$: $(\sqrt{2})^{2k} = 2^k$, so the sum is $2\sum {}^nC_{2k}2^k$

Every term is a whole number, so the sum is an integer for every positive integer n. A1

(c) Change the sign of $\sqrt{2}$ in (a): $(1-\sqrt{2})^5 = 41 - 29\sqrt{2}$

$|1-\sqrt{2}| < 1$, so its fifth power is tiny: $41 - 29\sqrt{2} = -0.0122\ldots$ M1

$\frac{41}{29} - \sqrt{2} = \frac{41-29\sqrt{2}}{29} = \frac{-0.0122\ldots}{29} = -0.00042\ldots$, and $0.00042 < 0.001$ ✓ A1

(Total for Question 8 is 8 marks)

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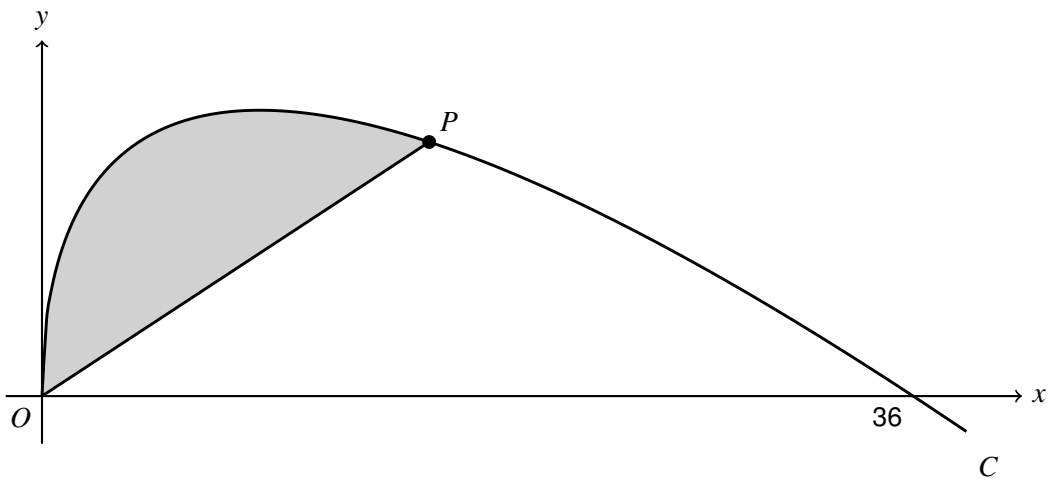

Figure 4

In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

Figure 4 shows part of the curve C with equation

$$y = 6\sqrt{x} - x \qquad x \geqslant 0$$

The curve meets the x-axis at O and at the point where $x = 36$

The point P, with x coordinate p, where $0 < p < 36$, lies on C.

The region shown shaded in Figure 4 is bounded by C and the straight line OP.

(a) Find, in terms of p, the area of the shaded region, giving your answer in its simplest form.

(5)

(b) Find the value of p for which the area of the shaded region is one eighth of the area of the region bounded by C and the x-axis.

(2)

(c) Show that, for this value of p, the point P is the maximum point of C.

(2)

(a) $\int (6x^{1/2} - x)\,dx = 6 \times \frac{2}{3}x^{3/2} - \frac{x^2}{2} = 4x^{3/2} - \frac{x^2}{2}$ M1

Area under C from 0 to p: $4p^{3/2} - \frac{p^2}{2}$ A1

Under the chord OP is a triangle, base p, height $6\sqrt{p} - p$: $\frac{1}{2}p(6\sqrt{p} - p) = 3p^{3/2} - \frac{p^2}{2}$ M1

Shaded = curve − triangle = $(4p^{3/2} - \frac{p^2}{2}) - (3p^{3/2} - \frac{p^2}{2})$ dM1

$= p^{3/2}$ A1

(b) Area under C from 0 to 36: $4(36)^{3/2} - \frac{36^2}{2} = 864 - 648 = 216$, so $p^{3/2} = \frac{216}{8} = 27$ M1

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Question 9 continued

$p = 27^{2/3} = 3^2$ $\boxed{p = 9}$ A1

(c) $\frac{dy}{dx} = 3x^{-1/2} - 1 = 0 \Rightarrow \sqrt{x} = 3 \Rightarrow x = 9$ M1

$\frac{d^2y}{dx^2} = -\frac{3}{2}x^{-3/2} < 0$, so a maximum at $x = 9 = p$: $P(9, 9)$ is the maximum point A1

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(Total for Question 9 is 9 marks)

TOTAL FOR PAPER IS 75 MARKS

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