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A-LEVEL MATHEMATICS • FURTHER PURE 2

Second-Order Differential Equations

Worked Solutions

Pearson Edexcel International A Level (WFM02) • Calculator allowed: show every step

Full worked solutions for the free Second-Order Differential Equations worksheet, A-Level and International A-Level Maths

A **Examples** & Your turn page 2

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Tap a section to jump straight to it

Full worked solutions for every question on the Further Pure 2 Second-Order Differential Equations worksheet, in the same order and with the same question numbers.

How to mark your work. Tick each step you got right, not just the final answer. At A-level most marks are for method, so a wrong final answer can still earn most of the marks. If you lost marks, find the *first* line where your working is different from ours. That is the thing to fix.

In Section E: **M** = method mark, **dM** = method mark that depends on the previous M mark, **A** = accuracy mark (needs the M mark before it), **B** = independent mark.

Question sheet | Exam questions by topic | Further Pure 2 exam questions | More worksheets

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A Your turn answers

Compare each line of your working with the solution.

CORE

A1 $m^2+m-12=0$, $m=3,-4$: $\boxed{y=Ae^{3x}+Be^{-4x}}$

A2 $(2m-1)^2=0$, $m=\frac{1}{2}$ twice: $\boxed{y=(A+Bx)e^{\frac{x}{2}}}$

A3 $m=\pm 4\text{i}$: $\boxed{y=A\cos 4x+B\sin 4x}$

STRETCH

A4 $-3a+2(ax+b)=6x+1$: $a=3$, $b=5$: $\boxed{y=3x+5}$

A5 $\lambda(9-4)=10$: $\boxed{y=2e^{3x}}$

A6 Only $\cos x$ is needed here (no first derivative term): $\lambda(-1+4)=15$: $\boxed{y=5\cos x}$

A7 $y=A\cos 3x+B\sin 3x$; $A=2$, $3B=6$: $\boxed{y=2\cos 3x+2\sin 3x}$

B Fluency solutions

One mark for each correct part. The notes in italics explain why each sequence was chosen.

CORE

B1 **(a)** $\boxed{Ae^{2x}+Be^{5x}}$ **(b)** $\boxed{(A+Bx)e^{-5x}}$ **(c)** $\boxed{e^x(A\cos 2x+B\sin 2x)}$ **(d)** $m=0,9$: $\boxed{A+Be^{9x}}$

The type of root decides the form.

B2 **(a)** $\boxed{2}$ **(b)** $-3a+2ax+2b=2x$: $\boxed{x+\frac{3}{2}}$ **(c)** $\lambda(9-9+2)=1$: $\boxed{\frac{1}{2}e^{3x}}$ **(d)** e^x is in the CF; try λxe^x: $\boxed{-xe^x}$

Check the CF before choosing the trial.

B3 **(a)** $\boxed{y = A\cos 2x + B\sin 2x + 2}$ **(b)** $\boxed{y = Ae^{x} + Be^{-x} + \frac{1}{3}e^{2x}}$

(c) $\boxed{y = e^{-x}(A\cos x + B\sin x) + 2x - 2}$ **(d)** $\boxed{y = (A + Bx)e^{3x} + 1}$

General solution = CF + PI, with two arbitrary constants.

B4 **(a)** $\boxed{y = Ae^{x} + Be^{-3x}}$ **(b)** $A + B = 4$, $A - 3B = 0$: $\boxed{y = 3e^{x} + e^{-3x}}$ **(c)** $\boxed{y = e^{x}}$ **(d)** $A = 0$: $\boxed{y = 2e^{-3x}}$

Two conditions fix two constants.

B5 **(a)** $\boxed{-3\cos x}$ **(b)** $\boxed{\sin x}$ **(c)** $-3p + 2q = 1$, $-2p - 3q = 0$: $\boxed{\frac{1}{13}(2\sin 2x - 3\cos 2x)}$

With a first-derivative term, both cos and sin are needed.

B6 **(a)** $\boxed{(A + Bx)e^{2x}}$ **(b)** $\lambda(1 - 4 + 4) = 1$: $\boxed{e^{x}}$ **(c)** try $\lambda x^{2}e^{2x}$, $2\lambda = 1$: $\boxed{\frac{1}{2}x^{2}e^{2x}}$ **(d)** $\boxed{2}$

A repeated root needs x^{2} for a clash.

STRETCH

B7 **(a)** $\boxed{A\cos 2x + B\sin 2x}$ **(b)** $\boxed{2x + e^{x}}$ **(c)** $\boxed{y = A\cos 2x + B\sin 2x + 2x + e^{x}}$ **(d)** $A + 1 = 1$, $2B + 3 = 3$: $\boxed{y = 2x + e^{x}}$

Treat each part of the right side separately.

B8 **(a)** $\boxed{x = A\cos 3t + B\sin 3t}$ **(b)** $\boxed{x = 2\cos 3t}$ **(c)** $\boxed{\frac{2\pi}{3}}$ **(d)** $\boxed{2}$

Pure imaginary roots give simple harmonic motion.

B9 **(a)** $\boxed{A\cos 3x + B\sin 3x}$ **(b)** it is the CF, so it gives 0 **(c)** $(-x\cos 3x)'' = 6\sin 3x + 9x\cos 3x$, so $(-x\cos 3x)'' + 9(-x\cos 3x) = 6\sin 3x$

(d) $\boxed{y = A\cos 3x + B\sin 3x - x\cos 3x}$

The resonant case grows with x

B10 **(a)** $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$ **(b)** differentiate again with the chain rule **(c)** $x^{2}\frac{d^{2}y}{dx^{2}} = \frac{d^{2}y}{dt^{2}} - \frac{dy}{dt}$, $x\frac{dy}{dx} = \frac{dy}{dt}$, giving the result

(d) $y = (A + Bt)e^{-t}$: $\boxed{y = \frac{A + B\ln x}{x}}$

The substitution gives constant coefficients.

C Reasoning solutions

Your wording can be different. Check that your reason makes the same point.

CORE

C1 His two terms combine into one, so there is only one constant: $\boxed{y = (A + Bx)e^{3x}}$

C2 The real part goes in the exponential, the imaginary part in the angles: $\boxed{y = e^{2x}(A\cos 3x + B\sin 3x)}$

C3 e^{2x} is in the CF, so it gives 0. Try $\lambda x e^{2x}$: $-\lambda = 1$: $\boxed{-xe^{2x}}$

C4 y' produces $\cos 2x$ terms, so both are needed: $-3p + 4q = 0$, $-4p - 3q = 5$: $\boxed{-\frac{4}{5}\cos 2x - \frac{3}{5}\sin 2x}$

STRETCH

C5 The conditions must be applied to the full solution: $y = Ae^{x} + Be^{-x} - 2$, $A + B = 2$, $A - B = 0$: $\boxed{y = e^{x} + e^{-x} - 2}$

C6 $e^{t} = x$, so $e^{2t} = x^2$: $\boxed{y = Ax + Bx^2}$

C7 A second-order equation has two arbitrary constants in its general solution; this has only one.

C8 **(a)** $y = xe^{mx}$: $y' = (1 + mx)e^{mx}$, $y'' = (2m + m^2x)e^{mx}$; $ay'' + by' + cy = e^{mx}\left(x(am^2 + bm + c) + (2am + b)\right) = 0$, since $am^2 + bm + c = 0$ and a repeated root has $m = -\dfrac{b}{2a}$

(b) $e^{(p+qi)x}$ is a solution, and since a, b, c are real its real part $e^{px}\cos qx$ is also a solution

D Problem solving solutions

There is often more than one correct method. Any complete, correct method scores full marks.

CORE

D1 CF $Ae^{3x} + Be^{-2x}$; e^{3x} clashes, so try $\lambda x e^{3x}$: $5\lambda = 10$: $\boxed{y = Ae^{3x} + Be^{-2x} + 2xe^{3x}}$

D2 CF $e^{x}(A\cos 2x + B\sin 2x)$; PI $2e^{x}$; $A + 2 = 0$, $A + 2B + 2 = 0$: $\boxed{y = 2e^{x}(1 - \cos 2x)}$

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D3 **(a)** Substituting gives $2\lambda e^{-x} = 4e^{-x}$: $\boxed{\lambda = 2}$

(b) $\boxed{y = (A + Bx)e^{-x} + 2x^2e^{-x}}$

D4 PI $2x - 2$; $y = Ae^{-x} + Be^{-2x} + 2x - 2$; $A + B = 2$, $-A - 2B + 2 = 0$: $\boxed{y = 2e^{-x} + 2x - 2}$

STRETCH

D5 **(a)** $m = -1 \pm 3i$; $x = e^{-t}(A\cos 3t + B\sin 3t)$, $A = 0$, $3B = 6$: $\boxed{x = 2e^{-t}\sin 3t}$

(b) $\dfrac{dx}{dt} = 2e^{-t}(3\cos 3t - \sin 3t) = 0$, $\tan 3t = 3$, $t = \dfrac{1}{3}\arctan 3$: $\boxed{x = 1.25}$

D6 **(a)** $x^2\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{dt^2} - \dfrac{dy}{dt}$ and $x\dfrac{dy}{dx} = \dfrac{dy}{dt}$, so the equation becomes $\dfrac{d^2y}{dt^2} + 4y = 0$, as required

(b) $y = A\cos 2t + B\sin 2t$, $t = \ln x$: $\boxed{y = A\cos(2\ln x) + B\sin(2\ln x)}$

D7 **(a)** PI $\lambda\cos t$: $3\lambda = 8$: $\boxed{x = A\cos 2t + B\sin 2t + \dfrac{8}{3}\cos t}$

(b) $A = -\dfrac{8}{3}$, $B = 0$: $\boxed{x = \dfrac{8}{3}(\cos t - \cos 2t)}$

D8 **(a)** PI $3x + 2$: $\boxed{y = (A + Bx)e^{3x} + 3x + 2}$

(b) $A = -2$, $B + 3A + 3 = 0$, $B = 3$: $\boxed{y = (3x - 2)e^{3x} + 3x + 2}$

(c) $\boxed{-5e^{-3} - 1}$

E Exam practice solutions

Mark-scheme notes are shown on the right of each line.

CORE *Exam questions for everyone*

E1 **(a)** $m^2 + m - 2 = 0$, $m = 1, -2$: CF $Ae^x + Be^{-2x}$ *M1 A1*

e^x is in the CF; PI λxe^x: $3\lambda = 6$ *M1*

$\boxed{y = Ae^x + Be^{-2x} + 2xe^x}$ *A1*

(b) $A + B = 1$, $A - 2B + 2 = 0$ *M1*

$\boxed{y = e^{-2x} + 2xe^x}$ *A1*

E2 **(a)** $m = -1 \pm 2\text{i}$: CF $\text{e}^{-x}(A\cos 2x + B\sin 2x)$ *M1 A1*

PI $p\cos x + q\sin x$: $4p + 2q = 10$, $4q - 2p = 0$, so $p = 2$, $q = 1$ *M1*

$\boxed{y = \text{e}^{-x}(A\cos 2x + B\sin 2x) + 2\cos x + \sin x}$ *A1*

(b) $A + 2 = 2$, $-A + 2B + 1 = 3$ *M1*

$\boxed{y = \text{e}^{-x}\sin 2x + 2\cos x + \sin x}$ *A1*

(c) The e^{-x} term dies away; y oscillates as $2\cos x + \sin x$, **amplitude $\sqrt{5}$** *B1*

STRETCH

E3 **(a)** $\dfrac{\text{d}y}{\text{d}x} = \dfrac{\text{d}y}{\text{d}t}\dfrac{\text{d}t}{\text{d}x} = \dfrac{1}{x}\dfrac{\text{d}y}{\text{d}t} = \text{e}^{-t}\dfrac{\text{d}y}{\text{d}t}$ *M1*

$\dfrac{\text{d}^2y}{\text{d}x^2} = \dfrac{\text{d}}{\text{d}t}\left(\text{e}^{-t}\dfrac{\text{d}y}{\text{d}t}\right)\dfrac{\text{d}t}{\text{d}x} = \text{e}^{-2t}\left(\dfrac{\text{d}^2y}{\text{d}t^2} - \dfrac{\text{d}y}{\text{d}t}\right)$, as required *A1**

(b) $x^2\dfrac{\text{d}^2y}{\text{d}x^2} = \dfrac{\text{d}^2y}{\text{d}t^2} - \dfrac{\text{d}y}{\text{d}t}$, $x\dfrac{\text{d}y}{\text{d}x} = \dfrac{\text{d}y}{\text{d}t}$, $\ln x = t$: (I) becomes (II), as required *M1 A1**

(c) $m = 2$ twice: CF $(A + Bt)\text{e}^{2t}$ *M1 A1*

PI $at + b$: $-4a + 4at + 4b = 2t$, $a = b = \dfrac{1}{2}$: $\boxed{y = (A + Bt)\text{e}^{2t} + \dfrac{1}{2}(t + 1)}$ *A1*

(d) $\text{e}^{2t} = x^2$, $t = \ln x$ *M1*

$\boxed{y = (A + B\ln x)x^2 + \dfrac{1}{2}(\ln x + 1)}$ *A1*