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A-LEVEL MATHEMATICS • FURTHER PURE 2

Second-Order Differential Equations

Worksheet

Pearson Edexcel International A Level (WFM02) • Calculator allowed: show every step

Free Second-Order Differential Equations worksheet for A-Level and International A-Level Maths, with full worked solutions

A — **Examples** & Your turn — page 2

B — **Fluency** core skills — page 4

C — **Reasoning** explain it — page 6

D — **Problem** solving — page 7

E — **Exam** practice — page 10

Tap a section to jump straight to it

By the end of this worksheet you can: solve the auxiliary equation • find the complementary function and a particular integral • *Stretch:* two conditions, clashes, and $x = e^t$

Which questions are for me?

CORE The methods every A-level student must secure. Start here.

STRETCH Every complementary function, clashing particular integrals, and $x = e^t$

Every section has both, clearly labelled. Questions get harder as you go, and every skill in Sections A to D is needed in Section E.

Track your progress

Section	B	C	D	E
Core marks	23	10	13	13
Stretch marks	16	14	17	9
My score				

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A Worked examples & Your turn

Read each example carefully. Cover it up, then answer the question beside it using the same method.

CORE *Secure the method*

Worked example A1: distinct roots

Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

$m^2 - 5m + 6 = 0$: $m = 2, 3$

$$y = Ae^{2x} + Be^{3x}$$

Your turn A1

Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 12y = 0$

Worked example A2: a repeated root

Solve $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$

$(m+3)^2 = 0$: $m = -3$ *twice*

$$y = (A + Bx)e^{-3x}$$

Your turn A2

Solve $4\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$

Worked example A3: complex roots

Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$

$m = -2 \pm 3i$

$$y = e^{-2x}(A\cos 3x + B\sin 3x)$$

Your turn A3

Solve $\frac{d^2y}{dx^2} + 16y = 0$

★★★★★

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STRETCH

Worked example A4: a polynomial PI

Find a particular integral of $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 4x$

Try $ax + b$: $-a - 2ax - 2b = 4x$

$a = -2$, $b = 1$

$y = 1 - 2x$

Your turn A4

Find a particular integral of $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 6x + 1$

Worked example A5: an exponential PI

Find a particular integral of $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 8e^x$

$\lambda(1 + 1 - 6)e^x = 8e^x$

$y = -2e^x$

Your turn A5

Find a particular integral of $\frac{d^2y}{dx^2} - 4y = 10e^{3x}$

Worked example A6: a trigonometric PI

Find a particular integral of $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 10\sin x$

Try $p\cos x + q\sin x$: $(p + 3q)\cos x + (q - 3p)\sin x$

$p + 3q = 0$, $q - 3p = 10$

$y = \sin x - 3\cos x$

Your turn A6

Find a particular integral of $\frac{d^2y}{dx^2} + 4y = 15\cos x$

Worked example A7: two conditions

Solve $\frac{d^2y}{dx^2} - 4y = 0$ given $y = 3$, $\frac{dy}{dx} = 2$ at $x = 0$

$A + B = 3$, $2A - 2B = 2$

$y = 2e^{2x} + e^{-2x}$

Your turn A7

Solve $\frac{d^2y}{dx^2} + 9y = 0$ given $y = 2$, $\frac{dy}{dx} = 6$ at $x = 0$

B Fluency

Each question is a designed sequence. From one part to the next, ask yourself: what changed, and what did it do to the answer?

CORE *Secure the method*

B1 Find the complementary function for

(a) $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 10y = 0$

(b) $\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 25y = 0$

(c) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0$

(d) $\frac{d^2y}{dx^2} - 9\frac{dy}{dx} = 0$

[4 marks]

B2 Find a particular integral of $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = f(x)$ when

(a) $f(x) = 4$

(b) $f(x) = 2x$

(c) $f(x) = e^{3x}$

(d) $f(x) = e^{x}$

[4 marks]

B3 Find the general solution of

(a) $\frac{d^2y}{dx^2} + 4y = 8$

(b) $\frac{d^2y}{dx^2} - y = e^{2x}$

(c) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 4x$

(d) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 9$

[4 marks]

B4 $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 0$

(a) Find the general solution.

(b) Find y given $y = 4$, $\frac{dy}{dx} = 0$ at $x = 0$

(c) Find y given $y = 1$, $\frac{dy}{dx} = 1$ at $x = 0$

(d) Find y given $y = 2$ at $x = 0$ and y bounded as $x \to \infty$

[4 marks]

B5 Find a particular integral of $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = f(x)$ when

(a) $f(x) = 3\sin x$

(b) $f(x) = \cos x$

(c) $f(x) = \cos 2x$

[3 marks]

B6 $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = f(x)$

(a) Find the complementary function.

(b) Find a PI when $f(x) = e^x$

(c) Find a PI when $f(x) = e^{2x}$

(d) Find a PI when $f(x) = 8$

[4 marks]

STRETCH

B7 $\frac{d^2y}{dx^2} + 4y = 8x + 5e^x$

(a) Find the complementary function.

(b) Find a particular integral.

(c) Write the general solution.

(d) Find y given $y = 1$, $\frac{dy}{dx} = 3$ at $x = 0$

[4 marks]

B8 $\frac{d^2x}{dt^2} + 9x = 0$ models an oscillation.

(a) Find the general solution.

(b) Find x given $x = 2$, $\frac{dx}{dt} = 0$ at $t = 0$

(c) State the period.

(d) State the amplitude.

[4 marks]

B9 $\frac{d^2y}{dx^2} + 9y = 6\sin 3x$

(a) Find the complementary function.

(b) Explain why $p\cos 3x + q\sin 3x$ fails as a PI.

(c) Show that $-x\cos 3x$ is a PI.

(d) Write the general solution.

[4 marks]

B10 $x = e^t$, $x > 0$

(a) Show that $\frac{dy}{dx} = e^{-t}\frac{dy}{dt}$

(b) Show that $\frac{d^2y}{dx^2} = e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$

(c) Show that $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = 0$ becomes $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0$

(d) Hence find y in terms of x

[4 marks]

C Reasoning

Show your working and explain your thinking. A correct answer without a reason does not score full marks.

CORE *Secure the method*

C1 The auxiliary equation has the repeated root $m = 3$. Amir writes $y = Ae^{3x} + Be^{3x}$. Explain his error, and correct it. **[2 marks]**

C2 The auxiliary equation has roots $2 \pm 3i$. Bea writes $y = e^{3x}(A\cos 2x + B\sin 2x)$. Explain her error, and correct it. **[2 marks]**

C3 For $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{2x}$, Cal tries λe^{2x} and finds $0 = e^{2x}$

Explain why, and find a particular integral. **[3 marks]**

C4 For $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 5\sin 2x$, Dev tries $q\sin 2x$

Explain why this fails, and find a particular integral. **[3 marks]**

STRETCH

C5 For $\frac{d^2y}{dx^2} - y = 2$ with $y = 0$ and $\frac{dy}{dx} = 0$ at $x = 0$, Eve applies the conditions to $Ae^x + Be^{-x}$ and then adds the PI, obtaining $y = -2$

Explain her error, and find the correct solution. **[3 marks]**

C6 After $x = e^t$, Fay obtains $y = Ae^t + Be^{2t}$ and writes $y = Ae^x + Be^{2x}$

Explain her error, and give y in terms of x **[3 marks]**

C7 Gus gives $y = Ae^{2x} + x$ as the general solution of a second-order differential equation.

Explain why this cannot be the general solution. **[3 marks]**

C8 $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, where $a \neq 0$

(a) Show that if the auxiliary equation has a repeated root m, then $y = xe^{mx}$ is a solution.

(3)

(b) Show that if the roots are $p \pm qi$, then $y = e^{px}\cos qx$ is a solution.

(2)

[5 marks]

D Problem solving

You decide the method. Read the whole question first, and show every step.

CORE *Secure the method*

D1 By first finding the complementary function, and choosing a suitable form for the particular integral, determine the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} - 6y = 10\mathrm{e}^{3x}$$

[3 marks]

D2 Determine the particular solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = 8\mathrm{e}^{x}$$

for which $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$, giving your answer in the form $y = \mathrm{f}(x)$ **[4 marks]**

D3 Given that $y = \lambda x^2\mathrm{e}^{-x}$ is a particular integral of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + y = 4\mathrm{e}^{-x}$$

(a) determine the value of the constant λ

(2)

(b) Hence determine the general solution of the differential equation.

(1)

[3 marks]

D4 Determine the particular solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 4x + 2$$

for which $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$, giving your answer in the form $y = \mathrm{f}(x)$ **[3 marks]**

STRETCH

D5 *In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.*

The displacement x metres of a particle at time t seconds satisfies

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\frac{\mathrm{d}x}{\mathrm{d}t} + 10x = 0$$

Given that $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 6$ when $t = 0$,

(a) determine x in terms of t

(2)

(b) determine, to 3 significant figures, the value of x at the first turning point for $t > 0$

(2)

[4 marks]

D6 **(a)** Given that $x = \mathrm{e}^t$, show that the differential equation

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 0, \quad x > 0$$

is transformed into $\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 4y = 0$

(2)

(b) Hence determine the general solution of the original equation, giving y in terms of x and simplifying your answer as far as possible.

(2)

[4 marks]

D7 The displacement x of a forced spring at time t satisfies

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4x = 8\cos t$$

(a) Determine the general solution of this differential equation, explaining your choice of particular integral.

(2)

(b) Given that the spring starts from rest with $x = 0$ when $t = 0$, determine x in terms of t

(2)

[4 marks]

D8 **(a)** Determine the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = 27x$$

(2)

(b) Given that $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ at $x = 0$, determine the particular solution of the differential equation.

(2)

(c) Hence find the value of y at $x = -1$, giving your answer in the form $pe^q + r$, where p, q and r are integers.

(1)

[5 marks]

E Exam practice

Exam-style questions in the style of WFM02. Write your answers in the spaces, and show every step.

CORE *Exam questions for everyone*

E1 **(a)** By first finding the complementary function, and explaining why λe^x cannot be used as the form of the particular integral, determine the general solution of the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 6e^x$$

(4)

Given that $y = 1$ and $\frac{dy}{dx} = 0$ when $x = 0$

(b) determine the particular solution of the differential equation, giving your answer in the form $y = f(x)$

(2)

(Total for Question E1 is 6 marks)

E2 **(a)** Determine the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 10\cos x$$

showing clearly how you find the complementary function and the particular integral.

(4)

Given that $y = 2$ and $\frac{dy}{dx} = 3$ when $x = 0$

(b) determine the particular solution of the differential equation.

(2)

(c) Describe the behaviour of y for large positive values of x, stating the amplitude of the resulting oscillation.

(1)

(Total for Question E2 is 7 marks)

STRETCH

E3 **(a)** Given that $x = e^t$, where $x > 0$, show that

$$\frac{dy}{dx} = e^{-t}\frac{dy}{dt} \quad \text{and} \quad \frac{d^2y}{dx^2} = e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

(2)

(b) Hence show that the transformation $x = e^t$ transforms the differential equation

$$x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2\ln x \quad \text{(I)}$$

into the differential equation

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 2t \quad \text{(II)}$$

(2)

(c) Solve differential equation (II) to determine a general solution for y in terms of t

(3)

(d) Hence determine the general solution of differential equation (I), simplifying any exponentials.

(2)

(Total for Question E3 is 9 marks)

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