



eClassroom

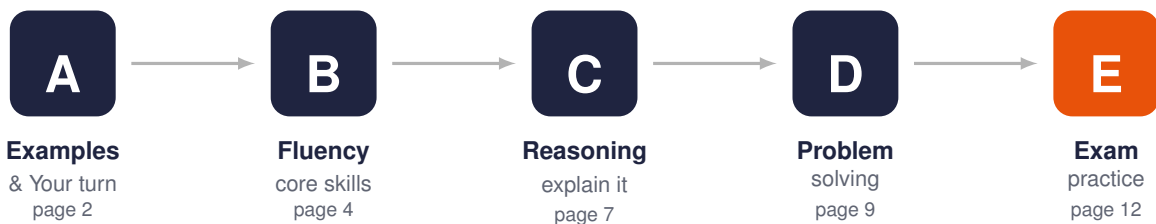
A-LEVEL MATHEMATICS • PURE 3

Numerical Methods

Worksheet

Pearson Edexcel International A Level (WMA13) & A Level (9MA0) • Calculator allowed: show every step

Free Numerical Methods worksheet for A-Level and International A-Level Maths, with full worked solutions



Tap a section to jump straight to it

By the end of this worksheet you can: change of sign • rearranging and iterating • *Stretch:* accuracy, convergence and roots from calculus.

Which questions are for me?

CORE The methods every A-level student must secure. Start here.

STRETCH Sign changes, rearrangements, iteration and roots to a given accuracy, with curves and models.

Every section has both, clearly labelled. Questions get harder as you go, and every skill in Sections A to D is needed in Section E.

Track your progress

Section	B	C	D	E
Core marks	23	10	13	11
Stretch marks	16	14	17	8
My score				

Full worked solutions

Exam questions by topic

Pure 3 exam questions

Formula sheet builder

Name

Class

Date



Join students already acing their exams.

Book your FREE lesson at www.eclassroom.org.uk

A**Worked examples & Your turn**

Read each example carefully. Cover it up, then answer the question beside it using the same method.

CORE*Secure the method***Worked example A1**

Show that $f(x) = x^3 - 2x - 5$ has a root in $[2, 3]$

$$f(2) = -1 < 0, f(3) = 16 > 0$$

a sign change, and f is continuous

so there is a root in $[2, 3]$

Your turn A1

Show that $f(x) = e^x - 3x$ has a root in $[1, 2]$

Worked example A2

Show that $x^3 - 2x - 5 = 0$ can be written as $x = \sqrt[3]{2x+5}$

Keep x^3 on one side, then take the cube root.

$$x^3 = 2x + 5$$

$$x = \sqrt[3]{2x+5}$$

Your turn A2

Show that $e^x - 3x = 0$ can be written as $x = \ln 3x$

Worked example A3

Use $x_{n+1} = \sqrt[3]{2x_n+5}$ with $x_1 = 2$ to find x_2 and x_3 to 4 d.p.

Substitute, then use the ANS key; do not round between steps.

$$x_2 = \sqrt[3]{9}$$

$$x_2 = 2.0801, \quad x_3 = 2.0924$$

Your turn A3

Use $x_{n+1} = \ln 3x_n$ with $x_1 = 1.5$ to find x_2 and x_3 to 4 d.p.



STRETCH

Worked example A4: a root to 3 d.p.

Show that the root of $x^3 - 2x - 5 = 0$ is 2.095 to 3 d.p.

Use $[2.0945, 2.0955]$: $f(2.0945) = -0.0006 < 0$,
 $f(2.0955) = 0.0106 > 0$

sign change, continuous: $\alpha = 2.095$

Your turn A4

Show that the root of $e^x - 3x = 0$ in $[1, 2]$ is 1.512 to 3 d.p.

Worked example A5: a false sign change

$f(x) = \frac{1}{x-2}$. Explain why the sign change on $[1, 3]$ does not show a root.

$f(1) = -1$, $f(3) = 1$, but f is not continuous at $x = 2$

asymptote at $x = 2$; no root

Your turn A5

Explain why the sign change of $\tan x$ on $[1, 2]$ (radians) does not show a root.

Worked example A6: choosing the iteration

$x^3 - 6x + 2 = 0$ has a root in $[0, 1]$. Use $x_{n+1} = \frac{x_n^3 + 2}{6}$, $x_1 = 0.3$, to find x_2 and x_3

The root near 0.34 is found by this rearrangement.

$x_2 = 0.3378$, $x_3 = 0.3398$

Your turn A6

The same equation has a root in $[2, 3]$. Use $x_{n+1} = \sqrt{6 - \frac{2}{x_n}}$, $x_1 = 2$, to find x_2 and x_3

Worked example A7: a rearrangement with logs

Show that $xe^x = 2$ can be written as $x = \ln \frac{2}{x}$

$e^x = \frac{2}{x}$, then take $\ln$ of both sides.

$x = \ln \frac{2}{x}$

Your turn A7

Show that $x^2 + \ln x = 4$ can be written as $x = \sqrt{4 - \ln x}$



B**Fluency**

Each question is a designed sequence. From one part to the next, ask yourself: what changed, and what did it do to the answer?

CORE

Secure the method

B1 Show that each equation has a root in the interval given.

(a) $x^3 + x - 5 = 0$, $[1, 2]$

(b) $\cos x - x = 0$, $[0, 1]$

(c) $\ln x + x - 3 = 0$, $[2, 3]$

(d) $2^x - 3x = 0$, $[3, 4]$

[4 marks]

B2 $x^3 + x - 5 = 0$

(a) Show that it can be written as $x = \sqrt[3]{5 - x}$

(b) Show that it can be written as $x = \frac{5}{x^2 + 1}$

(c) Show that it can be written as $x = \sqrt{\frac{5}{x} - 1}$

(d) Use the formula in (a) with $x_1 = 1.5$ to find x_2 and x_3 to 4 d.p.

[4 marks]

B3 $x_{n+1} = 2 + \ln x_n$, $x_1 = 3$. Find, to 4 d.p.,

(a) x_2

(b) x_3

(c) x_4

(d) the root, by repeated iteration (3 d.p.)

[4 marks]

B4 $f(x) = x^3 + x - 5$ has a root $\alpha = 1.516$ to 3 d.p.

(a) Find $f(1.5155)$

(b) Find $f(1.5165)$

(c) Hence show that $\alpha = 1.516$ to 3 d.p.

(d) Explain why the interval $[1.5155, 1.5165]$ is used.

[4 marks]



CORE *continued: when signs mislead*

B5 Explain whether a sign change (or no sign change) shows a root in each case.

(a) $\frac{1}{x-3}$ on $[2, 4]$

(b) $(x-1)^2$ on $[0, 2]$

(c) $\tan x$ on $[1.5, 1.6]$

[3 marks]

B6 $e^{-x} = x$

(a) Show that there is a root in $[0.5, 0.6]$

(b) Use $x_{n+1} = e^{-x_n}$ with $x_1 = 0.5$ to find x_2 and x_3

(c) Find x_4

(d) By repeated iteration, find the root to 3 d.p.

[4 marks]

STRETCH

B7 The curves $y = \ln x$ and $y = 3 - x^2$ meet at one point P

(a) Show that the x coordinate of P lies in $[1, 2]$

(b) Show that it satisfies $x = \sqrt{3 - \ln x}$

(c) Use this with $x_1 = 1.5$ to find x_2 and x_3 to 4 d.p.

(d) By repeated iteration, find the x coordinate of P to 3 d.p.

[4 marks]

B8 $2 \sin x = x$ has a root in $[1.8, 2]$ (x in radians).

(a) Show this with a sign change.

(b) Use $x_{n+1} = 2 \sin x_n$, $x_1 = 2$, to find x_2 and x_3

(c) Find x_4 , and describe how the iterates approach the root.

(d) By repeated iteration, find the root to 3 d.p.

[4 marks]



STRETCH*continued: roots from calculus and context*

B9 $y = e^x - x^3$ has a stationary point with $0 < x < 1$

(a) Show that its x coordinate satisfies $x = \sqrt{\frac{e^x}{3}}$

(b) Use this with $x_1 = 1$ to find x_2

(c) Find x_3 and x_4

(d) By repeated iteration, find the x coordinate to 3 d.p.

[4 marks]

B10 The depth h metres of water in a tank satisfies $h^3 - 6h^2 + 9 = 0$, with $1 < h < 2$

(a) Show that there is a root in $[1, 2]$

(b) Show that the equation can be written as $h = \sqrt{\frac{h^3 + 9}{6}}$

(c) Use this with $h_1 = 1.5$ to find h_2 and h_3

(d) By repeated iteration, find the depth to the nearest centimetre.

[4 marks]



C**Reasoning**

Show your working and explain your thinking. A correct answer without a reason does not score full marks.

CORE

Secure the method

- C1** Sam writes “ $f(1) = -2$ and $f(2) = 3$, so there is a root between 1 and 2.”
Explain what is missing from Sam’s answer. [2 marks]
- C2** $f(x) = \frac{2x}{x-1}$. Lily finds $f(0.5) = -2$ and $f(1.5) = 6$ and says there is a root between 0.5 and 1.5
Explain Lily’s mistake. [2 marks]
- C3** Kai uses $x_{n+1} = \frac{1}{2} + \cos x_n$, $x_1 = 1$, but his calculator is in degree mode.
Explain why Kai’s iterates are wrong, and find the correct x_2 to 4 d.p. [3 marks]
- C4** Jo uses $x_{n+1} = \sqrt[3]{2x_n + 5}$, $x_1 = 2$, rounds each iterate to 2 d.p. before using it, and writes $x_2 = 2.08$
Explain why Jo would lose marks. [3 marks]



STRETCH

- C5** To show that a root is 2.095 to 3 d.p., Max finds the signs of $f(2.094)$ and $f(2.096)$
Explain why Max's interval is not suitable, and state the interval he should use. [3 marks]
- C6** Ali uses $x_{n+1} = \frac{x_n^3 - 5}{2}$, $x_1 = 2$, for the root of $x^3 - 2x - 5 = 0$ near 2.09, and gets 1.5, -0.8125 , -2.7682
Explain what has happened, and suggest a better formula. [3 marks]
- C7** Kai says $f(x) = (x - 2)^2$ has no root in $[1, 3]$, because $f(1)$ and $f(3)$ are both positive.
Explain Kai's mistake. [3 marks]
- C8** (a) Show that a root α of $x^3 - 2x - 5 = 0$ satisfies $\alpha = g(\alpha)$, where $g(x) = \sqrt[3]{2x + 5}$ (1)
(b) Show that $0 < g'(x) < 0.16$ for $2 \leq x \leq 3$ (2)
(c) Explain why this means that the iteration $x_{n+1} = g(x_n)$ converges quickly to α from $x_1 = 2$ (2)
[5 marks]



D

Problem solving

You decide the method. Read the whole question first, and show every step.

CORE

Secure the method

- D1** The function f is defined by $f(x) = x^2 - 4 \sin x$, where x is measured in radians. The graph of $y = f(x)$ passes through the origin, and the equation $f(x) = 0$ has a single positive root α . You should make sure that your calculator is in radian mode throughout this question.

(a) Show that $1.9 < \alpha < 2$, stating clearly the condition and the conclusion that your argument needs.

(2)

(b) Show that α is a solution of $x = 2\sqrt{\sin x}$, explaining why the positive square root is taken.

(1)

[3 marks]

- D2** Figure 1 shows the curves with equations $y = e^{\frac{x}{2}}$ and $y = 4 - x^2$. They meet at exactly two points: P , which has a positive x coordinate, and Q , which has a negative x coordinate. The x coordinates of P and Q are roots of the equation $e^{\frac{x}{2}} + x^2 - 4 = 0$

(a) Show that the x coordinate of P lies between 1.4 and 1.5

(2)

(b) Using the iteration formula $x_{n+1} = \sqrt{4 - e^{\frac{x_n}{2}}}$ with $x_1 = 1.4$, find x_2 and x_3 , giving each to 4 decimal places and showing the substitution for x_2

(2)

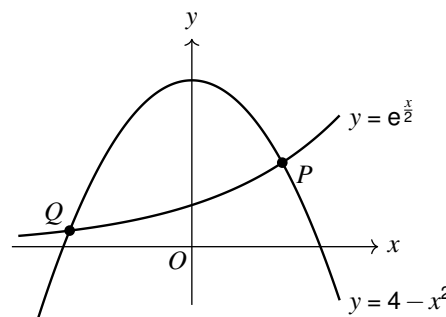


Figure 1

[4 marks]

- D3** A liquid is heated and then left to cool in a room. Its temperature, T °C, t minutes after it starts to cool is modelled by the equation $T = 20 + 70e^{-0.04t} - 0.5t$, for $0 \leq t \leq 30$. According to the model, the liquid first reaches a temperature of 40°C at time $t = \beta$ minutes.

(a) Show that β satisfies $t = 25 \ln \left(\frac{70}{20 + 0.5t} \right)$

(1)

(b) Use the iteration formula $t_{n+1} = 25 \ln \left(\frac{70}{20 + 0.5t_n} \right)$ with $t_1 = 20$ to find t_2 to 3 decimal places, and then find β to 3 significant figures by repeated iteration.

(2)

[3 marks]



- D4** The equation $x^3 + 3x - 7 = 0$ has exactly one real root, α . A student uses an iteration formula and finds that $\alpha = 1.406$, correct to 3 decimal places. You are asked to confirm this result without using further iteration. Using a suitable interval and the function $f(x) = x^3 + 3x - 7$, show that $\alpha = 1.406$ correct to 3 decimal places, giving the values of f that you use and a conclusion.

[3 marks]

STRETCH

- D5** The function f is defined by $f(x) = x^4 - 3x - 5$, $x \in \mathbb{R}$. The equation $f(x) = 0$ has exactly one positive root, α , and one negative root. In this question you must show the values that you use and give each conclusion clearly.
- (a) Show that α lies in the interval $[1.5, 2]$ (1)
- (b) Show that $f(x) = 0$ can be written as $x = \sqrt[4]{3x+5}$, and use $x_{n+1} = \sqrt[4]{3x_n+5}$ with $x_1 = 1.5$ to find x_2 and x_3 to 4 d.p. (2)
- (c) Find α to 4 decimal places by repeated iteration. (1)

[4 marks]



- D6** Figure 2 shows the curve with equation $y = e^{-x} + \sin x$, $0 \leq x \leq \pi$, where x is in radians. The curve has a single turning point, the maximum point M , whose x coordinate cannot be found exactly.

(a) Show that the x coordinate of M satisfies $x = \arccos(e^{-x})$

(2)

(b) Using this as an iteration formula with $x_1 = 1$, find x_2 and x_3 to 4 decimal places, and then find the x coordinate of M to 3 decimal places by repeated iteration.

(2)

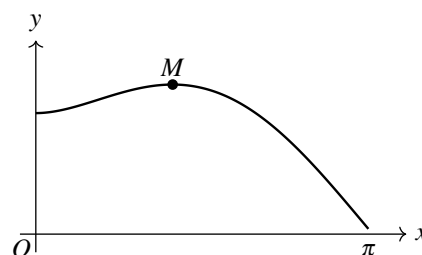


Figure 2

[4 marks]

- D7** The function f is defined by $f(x) = \frac{3}{2-x} + x$, $x \in \mathbb{R}$, $x \neq 2$. A student looks for a root of $f(x) = 0$ by checking the sign of $f(x)$ at the ends of the interval $[1.5, 2.5]$

(a) Show that $f(1.5)$ and $f(2.5)$ have opposite signs.

(1)

(b) Explain why this does not show that $f(x) = 0$ has a root between 1.5 and 2.5

(1)

(c) Find the roots of $f(x) = 0$ exactly, and show that neither lies in $[1.5, 2.5]$

(2)

[4 marks]

- D8** The function f is defined by $f(x) = x^3 - 7x + 3$, $x \in \mathbb{R}$. The equation $f(x) = 0$ has three real roots, one of which is negative. In this question you will find the two positive roots by iteration, using two different rearrangements of the equation.

(a) Show that $f(x) = 0$ has a root in $[0, 1]$ and a root in $[2, 3]$

(2)

(b) Use $x_{n+1} = \frac{x_n^3 + 3}{7}$ with $x_1 = 0.5$ to find x_2 and x_3 to 4 d.p.

(1)

(c) Show that the same formula with $x_1 = 2.5$ does not approach the root in $[2, 3]$, and use $x_{n+1} = \sqrt{7 - \frac{3}{x_n}}$ with $x_1 = 2.5$ to find that root to 3 d.p.

(2)

[5 marks]



E**Exam practice**

Exam-style questions in the style of WMA13. Write your answers in the spaces, and show every step.

CORE

Exam questions for everyone

E1 The function f is defined by

$$f(x) = 4\cos x + x^2 - 3 \quad 0 < x < \frac{\pi}{2}$$

where x is measured in radians. The equation $f(x) = 0$ has a single root α in this interval. You should make sure that your calculator is in radian mode throughout this question.

(a) Show that $1.1 < \alpha < 1.2$, explaining your reasoning clearly.

(2)

(b) Show that α is a solution of $x = \arccos\left(\frac{3-x^2}{4}\right)$

(1)

The iterative formula $x_{n+1} = \arccos\left(\frac{3-x_n^2}{4}\right)$ is used to find α

(c) Starting with $x_1 = 1.1$, find, to 4 decimal places, the value of x_2 , and find, by repeated iteration, the value of α to 3 decimal places.

(2)

(Total for Question E1 is 5 marks)



- E2** Figure 3 shows part of the curve C with equation $y = 4xe^{-x}$, $x \geq 0$, and the line with equation $y = 1$. The curve has a maximum point where $x = 1$. The line meets C at the two points A and B , where the x coordinate of B is β and $\beta > 1$

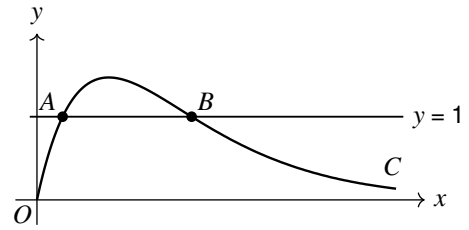


Figure 3

- (a) Show that β satisfies the equation $x = \ln 4x$

(1)

- (b) Show that $2 < \beta < 2.5$

(2)

- (c) Using $x_{n+1} = \ln 4x_n$ with $x_1 = 2$, find x_2 and x_3 to 4 decimal places.

.....

(2)

- (d) Explain why this iteration formula cannot be used to find the x coordinate of A , using the value $x_1 = 0.3$ to support your answer.

(1)

(Total for Question E2 is 6 marks)



STRETCH

E3 The path of a ball is modelled by

$$h = 1.2x - 0.4xe^{0.03x} \quad 0 \leq x \leq d$$

where h metres is the height of the ball above horizontal ground when it has travelled a horizontal distance of x metres. Figure 4 shows the path of the ball, which is modelled as a particle. The ball lands on the ground at D , after travelling a horizontal distance of d metres, and is at its greatest height at the point M

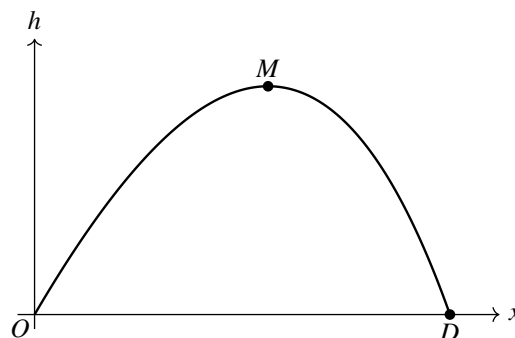


Figure 4

(a) Find the exact value of d , and give it to 2 decimal places.

.....
(2)

(b) Show that the x coordinate of M satisfies $x = \frac{100}{3} \ln \left(\frac{3}{1 + 0.03x} \right)$

(3)

(c) Using the iteration formula from (b) with $x_1 = 20$, find x_2 to 3 decimal places, and, by repeated iteration, the x coordinate of M to 2 decimal places.

.....
(2)

(d) Hence find the greatest height of the ball.

.....
(1)

(Total for Question E3 is 8 marks)

Want more exam practice? The **Exam Questions by Topic** booklets have 10 more full exam questions on Pure 3 Chapter 8, with a full mark scheme.

[Open the booklet](#)

Finished? Mark your work with the **full worked solutions**, fill in your scores on the front page, then practise real exam questions on the **Pure 3 exam questions page**.



Join students already acing their exams.

Book your FREE lesson at www.eclassroom.org.uk