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A-LEVEL MATHEMATICS • FURTHER PURE 1

Numerical Solutions of Equations

Worksheet

Pearson Edexcel International A Level (WFM01) • Calculator allowed: show every step

Free Numerical Solutions of Equations worksheet for A-Level and International A-Level Maths, with full worked solutions

A **Examples** & Your turn page 2

B **Fluency** core skills page 4

C **Reasoning** explain it page 6

D **Problem** solving page 7

E **Exam** practice page 10

Tap a section to jump straight to it

By the end of this worksheet you can: locate a root by a change of sign • use interval bisection • use linear interpolation • use the Newton–Raphson process • *Stretch:* fractional and negative powers, accuracy checks, and the limitations of each method.

Which questions are for me?

CORE The methods every A-level student must secure. Start here.

STRETCH Fractional powers in Newton–Raphson, methods that fail, and accuracy checks with a conclusion.

Every section has both, clearly labelled. Questions get harder as you go, and every skill in Sections A to D is needed in Section E.

Track your progress

Section	B	C	D	E
Core marks	23	10	13	13
Stretch marks	16	14	17	9
My score				

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Name ..... Class ..... Date .....

A Worked examples & Your turn

Read each example carefully. Cover it up, then answer the question beside it using the same method.

CORE *Secure the method*

Worked example A1: a change of sign

Show that $f(x) = x^3 - 4x - 2$ has a root in $[2,3]$

$f(2) = -2 < 0$, $f(3) = 13 > 0$

sign change, f continuous, so a root in $[2,3]$

Your turn A1

Show that $f(x) = x^3 + 2x - 4$ has a root in $[1,2]$

Worked example A2: interval bisection

$f(x) = x^2 - 7$ has a root in $[2,3]$. Use interval bisection twice.

$f(2.5) = -0.75$: root in $[2.5,3]$; $f(2.75) = 0.5625$: root in $[2.5,2.75]$

[2.5, 2.75]

Your turn A2

$f(x) = x^3 - 2x - 5$ has a root in $[2,3]$. Use interval bisection twice.

Worked example A3: linear interpolation

$f(x) = x^3 - 3x - 1$ has a root in $[1,2]$. Use linear interpolation once.

$f(1) = -3$, $f(2) = 1$; *similar triangles*

$$\frac{x-1}{3} = \frac{2-x}{1}$$

$x = 1.75$

Your turn A3

$f(x) = x^3 + 2x - 7$ has a root in $[1,2]$. Use linear interpolation once, to 3 decimal places.

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STRETCH

Worked example A4: Newton–Raphson

Apply Newton–Raphson once to $f(x) = x^3 - 2x - 5$ with $x_0 = 2$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$f(2) = -1$, $f'(x) = 3x^2 - 2$, $f'(2) = 10$

$x_1 = 2.1$

Your turn A4

Apply Newton–Raphson once to $f(x) = x^3 + 3x - 1$ with $x_0 = 0.5$, to 3 decimal places.

Worked example A5: fractional powers

Apply Newton–Raphson once to $f(x) = 2\sqrt{x} - x + 1$ with $x_0 = 5$, to 3 decimal places.

$f'(x) = x^{-\frac{1}{2}} - 1$; $f(5) = 0.4721$, $f'(5) = -0.5528$

$x_1 = 5.854$

Your turn A5

Apply Newton–Raphson once to $f(x) = \frac{3}{x} - x + 2$ with $x_0 = 2.5$, to 3 decimal places.

Worked example A6: an accuracy check

Show that the root of $x^3 - 2x - 5 = 0$ near 2.1 is 2.09 to 2 decimal places.

Test the bounds 2.085 and 2.095

$f(2.085) = -0.106$, $f(2.095) = 0.005$: sign change

so the root is 2.09 to 2 d.p.

Your turn A6

Show that the positive root of $x^2 - 7 = 0$ is 2.646 to 3 decimal places.

Worked example A7: when Newton–Raphson fails

Explain why $x_0 = 1$ cannot be used for $f(x) = x^3 - 3x - 1$

$f'(x) = 3x^2 - 3$, so $f'(1) = 0$

cannot divide by $f'(1) = 0$

Your turn A7

Explain why $x_0 = 2$ cannot be used for $f(x) = x^2 - 4x + 1$

B Fluency

Each question is a designed sequence. From one part to the next, ask yourself: what changed, and what did it do to the answer?

CORE *Secure the method*

B1 $f(x) = x^3 - 5x + 1$. State, with a reason, whether $f(x) = 0$ must have a root in

(a) $[0, 1]$

(b) $[1, 2]$

(c) $[2, 3]$

(d) $[-3, -2]$

[4 marks]

B2 $f(x) = x^3 - 5x + 1$ has a root in $[2, 3]$. Find the interval after

(a) one bisection

(b) two bisections

(c) three bisections

(d) How many bisections give an interval of width less than 0.01?

[4 marks]

B3 $f(x) = x^2 - 10$. Use linear interpolation once on each interval, giving answers to 4 decimal places.

(a) $[3, 4]$

(b) $[3.1, 3.2]$

(c) $[3.16, 3.17]$

(d) Compare each answer with $\sqrt{10}$ and comment.

[4 marks]

B4 Find $f'(x)$ for

(a) $x^4 - 3x$

(b) $2\sqrt{x}$

(c) $\dfrac{4}{x^2}$

(d) $x^{\frac{3}{2}} - \dfrac{5}{x}$

[4 marks]

B5 $f(x) = x^2 - 5$, with $x_0 = 2$

(a) Find x_1 using Newton–Raphson.

(b) Find x_2, to 4 decimal places.

(c) Compare x_2 with $\sqrt{5}$

[3 marks]

B6 $f(x) = x^3 + x - 3$ has a root α in $[1, 2]$

(a) Find f(1.2)

(b) Find f(1.25)

(c) Show that $\alpha = 1.213$ to 3 decimal places.

(d) Explain why finding f(1.213) alone would not show this.

[4 marks]

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STRETCH

B7 $f(x) = x^2 + \frac{1}{x} - 4,\ x > 0$

(a) Find $f'(x)$

(b) Use $x_0 = 2$ to find x_1, to 4 decimal places.

(c) Find x_2, to 4 decimal places.

(d) Show that the root is 1.86 to 2 decimal places.

[4 marks]

B8 Explain what goes wrong in each case.

(a) Newton–Raphson with $x_0 = 1$ for $x^3 - 3x + 1 = 0$

(b) $\tan x$ changes sign on $[1, 2]$ (radians)

(c) One interpolation on $[0, 2]$ for $x^2 - 2 = 0$ gives 1

(d) Newton–Raphson with $x_0 = 0.9$ for $x^3 - 3x + 1 = 0$ gives $x_1 = -0.80$

[4 marks]

B9 $f(x) = x^4 - 5x + 2$

(a) Show that $f(x) = 0$ has a root in $[0, 1]$

(b) Show that it has another root in $[1, 2]$

(c) Use linear interpolation once on $[0, 1]$

(d) Use Newton–Raphson once from 0.5

[4 marks]

B10 $f(x) = x^3 + 2x - 20$ has a root $\alpha = 2.4695$ to 4 d.p.

(a) Find f(2) and f(3)

(b) Use linear interpolation once on $[2, 3]$

(c) Use interval bisection once on $[2, 3]$

(d) Which method gave the better estimate here?

[4 marks]

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C Reasoning

Show your working and explain your thinking. A correct answer without a reason does not score full marks.

CORE *Secure the method*

C1 To show that $f(x) = x^3 - x - 1$ has a root in $[1, 2]$, Amir writes "$f(1) = -1$, $f(2) = 5$". Explain why his answer is incomplete, and complete it. **[2 marks]**

C2 Bea uses $x_1 = x_0 + \dfrac{f(x_0)}{f'(x_0)}$ for $f(x) = x^2 - 3$ with $x_0 = 2$ and gets 2.25. Explain her error, and find x_1 **[2 marks]**

C3 $f(x) = x - \dfrac{4}{\sqrt{x}}$. Cal writes $f'(x) = 1 - 2x^{-\frac{1}{2}}$

Explain his error, find $f'(x)$, and apply Newton–Raphson once with $x_0 = 2.5$ **[3 marks]**

C4 To show that the positive root of $x^2 - 3 = 0$ is 1.73 to 2 decimal places, Dev finds $f(1.73) = -0.0071$ only.

Explain why this is not enough, and complete the check. **[3 marks]**

STRETCH

C5 $f(x) = 1 + \dfrac{1}{x - 3}$. Eve finds $f(2.5) = -1$ and $f(3.5) = 3$, and says there is a root in $[2.5, 3.5]$

Explain her error, and find the actual root of $f(x) = 0$ **[3 marks]**

C6 Fay tries Newton–Raphson with $x_0 = 2$ for $f(x) = x^3 - 12x + 5$

Explain why this fails, and suggest a better starting value for the root in $[3, 4]$ **[3 marks]**

C7 Gus says that one linear interpolation on $[1, 3]$ for $x^2 - 2 = 0$ gives the root.

Find the interpolation estimate, and explain why it is not the root. **[3 marks]**

C8 The curve $y = f(x)$ has a tangent at the point $(x_0, f(x_0))$

(a) Show that this tangent meets the x-axis at $x = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

(3)

(b) Use this to explain why the Newton–Raphson process fails when $f'(x_0) = 0$

(2)

[5 marks]

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D Problem solving

You decide the method. Read the whole question first, and show every step.

CORE *Secure the method*

D1 *In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.*

$f(x) = x^3 + 4x - 2$

Given that the equation $f(x) = 0$ has a single root, α, in the interval $[0, 1]$, use interval bisection to determine an interval of width 0.25 that contains α. You should show the value of $f(x)$ at each midpoint you use, and state clearly the interval that contains α after each step. **[3 marks]**

D2 *In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.*

$f(x) = x^2 - 3\sqrt{x} - 1, \ x > 0$

(a) Show that the equation $f(x) = 0$ has a root α in the interval $[2, 3]$, giving a reason for your answer.

(1)

(b) Determine $f'(x)$, writing each term as a power of x before differentiating.

(1)

(c) Using $x_0 = 2.5$ as a first approximation to α, apply the Newton–Raphson process once to $f(x)$ to determine a second approximation to α, giving your answer to 3 decimal places.

(2)

[4 marks]

D3 $g(x) = 2\sin x + x - 2$, where x is in radians.

(a) Show that the equation $g(x) = 0$ has a root, β, in the interval $[0.5, 1]$, explaining your reasoning fully.

(1)

(b) Use linear interpolation once on the interval $[0.5, 1]$ to determine an approximation for β, giving your answer to 3 significant figures. You must show the values of g(0.5) and g(1) that you use.

(2)

[3 marks]

D4 $f(x) = x^3 - 6x^2 + 9x - 1$

The equation $f(x) = 0$ has three real roots, one of which, α, is close to zero.

(a) Show that α lies in the interval $[0, 0.5]$, giving a reason for your answer.

(1)

(b) Using $x_0 = 0.2$ as a first approximation to α, and showing the values of f(0.2) and $f'(0.2)$, apply the Newton–Raphson process once to $f(x)$ to determine a second approximation to α, giving your answer to 3 decimal places.

(2)

[3 marks]

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STRETCH

D5 $f(x) = x^2 - \frac{5}{x} - 3, \ x > 0$

The equation $f(x) = 0$ has a single positive root α in the interval $[2, 3]$, which is to be found using the Newton–Raphson process.

(a) Determine $f'(x)$, writing $\frac{5}{x}$ as a power of x first.

(2)

(b) Using $x_0 = 2.5$ as a first approximation to α, apply the Newton–Raphson process once to $f(x)$ to determine a second approximation to α, giving your answer to 3 decimal places.

(2)

[4 marks]

D6 $f(x) = x^3 + x - 5$

The equation $f(x) = 0$ has a single root α in the interval $[1, 2]$. Two different numerical methods are to be used to estimate α

(a) Starting with the interval $[1, 2]$, use interval bisection twice to show that α lies in the interval $[1.5, 1.75]$

(2)

(b) Use linear interpolation once on the interval $[1.5, 1.75]$ to obtain an approximation for α, giving your answer to 3 decimal places.

(2)

[4 marks]

D7 $f(x) = x^3 - 4x + 1$

(a) Find the values of $f(x)$ for $x = -3, -2, -1, 0, 1, 2$ and hence, explaining your reasoning, determine intervals of width one that contain the three roots of $f(x) = 0$

(2)

(b) Use linear interpolation once on the interval containing the negative root to find an approximation for that root, giving your answer to 3 significant figures.

(2)

[4 marks]

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D8 *In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.*

$f(x) = x^3 - 4\sqrt{x} - 3, \ x > 0$

The equation $f(x) = 0$ has a single root α in the interval $[2, 2.5]$. Two different methods are to be used to find approximations to α

(a) Use linear interpolation once on the interval $[2, 2.5]$ to determine an approximation to α, giving your answer to 3 decimal places.

(2)

(b) Determine $f'(x)$

(1)

(c) Using $x_0 = 2$ as a first approximation to α, apply the Newton–Raphson process once to $f(x)$ to determine a second approximation to α, giving your answer to 3 decimal places.

(2)

[5 marks]

E Exam practice

Exam-style questions in the style of WFM01. Write your answers in the spaces, and show every step.

CORE *Exam questions for everyone*

E1 *In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.*

$$g(x) = \frac{1}{2}x^2 - \frac{4}{\sqrt{x}} - 5, \quad x > 0$$

The equation $g(x) = 0$ has exactly one root, β, which is positive.

(a) Show that β lies in the interval $[3, 4]$, explaining your reasoning.

(2)

(b) Determine $g'(x)$, giving each term in its simplest form.

(2)

(c) Using $x_0 = 3.5$ as a first approximation to β, apply the Newton–Raphson procedure once to $g(x)$ to find a second approximation to β, giving your answer to 3 decimal places.

(2)

(Total for Question E1 is 6 marks)

E2 *In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.*

$f(x) = x^3 - 7x + 3$

The equation $f(x) = 0$ has three real roots. One of them, α, lies between 0 and 1

(a) Show that the equation $f(x) = 0$ has a root α in the interval $[0, 1]$, explaining your reasoning.

(2)

(b) Starting with the interval $[0, 1]$, use interval bisection twice to determine an interval of width 0.25 that contains α

(3)

(c) Use linear interpolation once on the interval found in part (b) to determine another approximation for α, giving your answer to 3 decimal places.

(2)

(Total for Question E2 is 7 marks)

STRETCH

E3 *In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.*

$f(x) = 2x - 6\sqrt{x} + 3, \quad x > 0$

(a) Determine $f'(x)$

(2)

(b) Explain why $x_0 = 2.25$ cannot be used as a first approximation to a root in the Newton–Raphson process for this function, showing any calculation you make.

(1)

The equation $f(x) = 0$ has a root α greater than 2.25

(c) Show that α lies in the interval $[5, 6]$

(2)

(d) Using $x_0 = 5.5$ as a first approximation to α, apply the Newton–Raphson process once to $f(x)$ to obtain a second approximation to α, giving your answer to 3 decimal places.

(2)

(e) Show that $\alpha = 5.60$ correct to 2 decimal places, stating clearly the values you calculate and your conclusion.

(2)

(Total for Question E3 is 9 marks)

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