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A-LEVEL MATHEMATICS • FURTHER PURE 2

Maclaurin and Taylor Series

Worked Solutions

Pearson Edexcel International A Level (WFM02) • Calculator allowed: show every step

Full worked solutions for the free Maclaurin and Taylor Series worksheet, A-Level and International A-Level Maths

A **Examples** & Your turn page 2

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Tap a section to jump straight to it

Full worked solutions for every question on the Further Pure 2 Maclaurin and Taylor Series worksheet, in the same order and with the same question numbers.

How to mark your work. Tick each step you got right, not just the final answer. At A-level most marks are for method, so a wrong final answer can still earn most of the marks. If you lost marks, find the *first* line where your working is different from ours. That is the thing to fix.

In Section E: **M** = method mark, **dM** = method mark that depends on the previous M mark, **A** = accuracy mark (needs the M mark before it), **B** = independent mark.

Question sheet | Exam questions by topic | Further Pure 2 exam questions | More worksheets

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A Your turn answers

Compare each line of your working with the solution.

CORE

A1 $f(0) = 1$, $f''(0) = -4$, $f''''(0) = 16$; odd ones 0: $\boxed{1 - 2x^2 + \frac{2}{3}x^4}$

A2 $(1+u)^{-1}$ with $u = -3x$: $\boxed{1 + 3x + 9x^2 + 27x^3,\ |x| < \frac{1}{3}}$

A3 $(1+x)\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6}\right)$: $\boxed{1 - \frac{1}{2}x^2 + \frac{1}{3}x^3}$

STRETCH

A4 $f(4) = 2$, $f'(4) = \frac{1}{4}$, $f''(4) = -\frac{1}{32}$, $f'''(4) = \frac{3}{256}$: $\boxed{2 + \frac{x-4}{4} - \frac{(x-4)^2}{64} + \frac{(x-4)^3}{512}}$

A5 $y'(0) = 1$; $y'' = y + xy'$ gives 0; $y''' = 2y' + xy''$ gives 2: $\boxed{y = x + \frac{1}{3}x^3}$

A6 $\dfrac{\frac{x^3}{6} - \frac{x^5}{120} + \cdots}{x^3}$: $\boxed{\frac{1}{6}}$

A7 $0.1 - \dfrac{0.01}{2} + \dfrac{0.001}{3}$: $\boxed{\mathbf{0.09533}}$

B Fluency solutions

One mark for each correct part. The notes in italics explain why each sequence was chosen.

CORE

B1 **(a)** $\boxed{1 - 2x + 2x^2 - \frac{4}{3}x^3}$ **(b)** $\boxed{3x - \frac{9}{2}x^3}$ **(c)** $\boxed{1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}}$ **(d)** $\boxed{1 + \frac{1}{2}x^2}$

Find each derivative at 0, then divide by n!

B2 **(a)** $1+x^2+\frac{x^4}{2}+\frac{x^6}{6}$, **all** x **(b)** $1-\frac{x^2}{8}+\frac{x^4}{384}$, **all** x

(c) $-3x-\frac{9}{2}x^2-9x^3, -\frac{1}{3}\leqslant x<\frac{1}{3}$ **(d)** $1-x+\frac{3}{2}x^2, |x|<\frac{1}{2}$

Substituting changes the range of validity.

B3 **(a)** $1-x+\frac{1}{3}x^3$ **(b)** $2x+\frac{2}{3}x^3+\frac{2}{5}x^5$ **(c)** $\frac{1}{2}(1+\cos 2x)$: $1-x^2+\frac{1}{3}x^4$ **(d)** $1+x+\frac{1}{2}x^2$ (no x^3 term)

Keep every term up to the required power in each factor.

B4 **(a)** $e^2\left(1+(x-2)+\frac{1}{2}(x-2)^2\right)$ **(b)** $\frac{1}{2}+\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)-\frac{1}{4}\left(x-\frac{\pi}{6}\right)^2$

(c) $\frac{1}{2}-\frac{x-2}{4}+\frac{(x-2)^2}{8}$ **(d)** **0.52269**

A Taylor series is accurate near its centre.

B5 **(a)** $y'=(x+1)e^x$, $y''=(x+2)e^x$, $y'''=(x+3)e^x$ **(b)** $x+x^2+\frac{1}{2}x^3+\frac{1}{6}x^4$ **(c)** $\frac{n}{n!}$: $\frac{1}{(n-1)!}$

A pattern in the derivatives gives the general term.

B6 **(a)** 2 **(b)** $-\frac{9}{2}$ **(c)** $\frac{1}{3}$ **(d)** 2

The lowest surviving power decides the limit.

STRETCH

B7 **(a)** -1 **(b)** $y'''=-2y'-xy''$: 0 **(c)** $y''''=-3y''-xy'''$: 3 **(d)** $y=1-\frac{1}{2}x^2+\frac{1}{8}x^4$

Differentiate the equation itself, not the series.

B8 **(a)** differentiate $1+y^2$ **(b)** $2\left(\frac{dy}{dx}\right)^2+2y\frac{d^2y}{dx^2}$ **(c)** $0, 1, 0, 2$ **(d)** $x+\frac{1}{3}x^3$

A relation makes the higher derivatives quick.

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B9 **(a)** $\ln 2 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{24}, -2 < x \leqslant 2$ **(b)** $\frac{1}{4} + \frac{x}{16} + \frac{x^2}{64}, |x| < 4$

(c) $3 + \frac{x}{6} - \frac{x^2}{216}, |x| < 9$ **(d)** **(b) and (c)**

Take out the constant first, then substitute.

B10 **(a)** **0.9800667** **(b)** $9.1 \times 10^{-6}\,\%$ **(c)** $0.5 - \frac{0.5^3}{6} + \frac{0.5^5}{120}$: **0.479427** **(d)** $\sin 0.5 = 0.479426$, very close

Small x gives a very accurate series.

C Reasoning solutions

Your wording can be different. Check that your reason makes the same point.

CORE

C1 The coefficient is $\frac{f''(0)}{2!}$: $\frac{9}{2}x^2$

C2 $\ln(1+u)$ needs $-1 < u \leqslant 1$ with $u = 2x$: $-\frac{1}{2} < x \leqslant \frac{1}{2}$

C3 Differentiating y^2 needs the chain rule, $2y\frac{dy}{dx}$, not $\left(\frac{dy}{dx}\right)^2$: $\frac{d^2y}{dx^2} = 2y\frac{dy}{dx} + 1$

C4 $\sin 0 = 0$, so $y(0) = 0$; $y'(0) = 1$: **the first term is x**

STRETCH

C5 $-2x\sin x - 2x\sin x = -4x\sin x$: $2\cos x - 4x\sin x - x^2\cos x$

C6 A Taylor series about 1 is in powers of $(x-1)$: $(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$

C7 $(1+x)^{-1}$ must go to x^2, since $x \times x^2$ gives x^3: $x - x^2 + \frac{5}{6}x^3$

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C8 **(a)** $e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \ldots$; grouping real and imaginary terms gives $\cos x + i\sin x$, as required

(b) $e^{i\pi} = \cos\pi + i\sin\pi = -1$, as required; $\boxed{e^{\frac{\pi}{2}i} = i}$

D Problem solving solutions

There is often more than one correct method. Any complete, correct method scores full marks.

CORE

D1 **(a)** $\boxed{\frac{2}{(4+x)^3}}$

(b) $\ln 4 + \frac{x}{4} - \frac{x^2}{32} + \frac{2}{64}\cdot\frac{x^3}{6}$: $\boxed{\ln 4 + \frac{x}{4} - \frac{x^2}{32} + \frac{x^3}{192}}$

D2 **(a)** $y' = e^{2x}(2\sin x + \cos x)$, $y'' = e^{2x}(3\sin x + 4\cos x) = 4y' - 5y$, as required

(b) $y(0) = 0$, $y'(0) = 1$, $y''(0) = 4$, $y'''(0) = 4(4) - 5(1) = 11$: $\boxed{x + 2x^2 + \frac{11}{6}x^3}$

D3 $y'(0) = 1$; $y'' = y + xy' + 2yy'$ gives 3; $y''' = 2y' + xy'' + 2(y')^2 + 2yy''$ gives 10: $\boxed{y = 1 + x + \frac{3}{2}x^2 + \frac{5}{3}x^3}$

D4 **(a)** $\boxed{\frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\left(x - \frac{\pi}{3}\right)^2}$

(b) $x - \frac{\pi}{3} = -0.04720$: $\boxed{\mathbf{0.5403}}$

STRETCH

D5 **(a)** $y' = 2\sec^2 2x = 4$, $y'' = 8\sec^2 2x\tan 2x = 16$, $y''' = 8\left(4\sec^2 2x\tan^2 2x + 2\sec^4 2x\right) = 128$ at $\frac{\pi}{8}$:

$\boxed{1 + 4\left(x - \frac{\pi}{8}\right) + 8\left(x - \frac{\pi}{8}\right)^2 + \frac{64}{3}\left(x - \frac{\pi}{8}\right)^3}$

(b) $x = 0.35$: $\boxed{\mathbf{0.8421}}$

D6 **(a)** differentiate: $y'y'' + yy''' + 2y'y'' = 0$, so $y''' = -\frac{3}{y}y'y''$, as required

(b) $y''(0) = \frac{2-1}{1} = 1$, $y'''(0) = -3$: $\boxed{y = 1 + x + \frac{1}{2}x^2 - \frac{1}{2}x^3}$

D7 **(a)** $y' = \dfrac{\cos x}{1+\sin x}$; $y'' = \dfrac{-\sin x(1+\sin x) - \cos^2 x}{(1+\sin x)^2} = \dfrac{-(1+\sin x)}{(1+\sin x)^2}$, as required

(b) $y(0) = 0$, $y'(0) = 1$, $y''(0) = -1$, $y''' = \dfrac{\cos x}{(1+\sin x)^2}$ gives 1: $\boxed{x - \frac{1}{2}x^2 + \frac{1}{6}x^3}$

D8 **(a)** $y''(1) = 3$; $y''' = y + xy'$: $\boxed{2}$; $y'''' = 2y' + xy''$: $\boxed{3}$

(b) $\boxed{y = 2 + \frac{3}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \frac{1}{8}(x-1)^4}$

E Exam practice solutions

Mark-scheme notes are shown on the right of each line.

CORE *Exam questions for everyone*

E1 **(a)** $y' = \dfrac{3}{2+3x}$, $y'' = -\dfrac{9}{(2+3x)^2}$ *M1*

$\boxed{\dfrac{d^3y}{dx^3} = \dfrac{54}{(2+3x)^3}}$ *A1*

(b) $y(0) = \ln 2$, $\dfrac{3}{2}$, $-\dfrac{9}{4}$, $\dfrac{27}{4}$ *M1*

$\boxed{\ln 2 + \frac{3}{2}x - \frac{9}{8}x^2 + \frac{9}{8}x^3}$ *A1*

(c) $\ln(2-3x) = \ln 2 - \dfrac{3}{2}x - \dfrac{9}{8}x^2 - \dfrac{9}{8}x^3$ *B1*

$\boxed{3x + \frac{9}{4}x^3}$ *A1*

E2 **(a)** $\dfrac{d^2y}{dx^2} = 2y\dfrac{dy}{dx} + 2$ *M1*

differentiate again: $2\left(\dfrac{dy}{dx}\right)^2 + 2y\dfrac{d^2y}{dx^2}$, as required *A1**

(b) At $x = 1$: $y' = 1 + 2 = 3$, $y'' = 2(1)(3) + 2 = 8$, $y''' = 2(9) + 2(1)(8) = 34$ *M1 A1*

$y = 1 + 3(x-1) + \dfrac{8}{2}(x-1)^2 + \dfrac{34}{6}(x-1)^3$ *M1*

$\boxed{y = 1 + 3(x-1) + 4(x-1)^2 + \frac{17}{3}(x-1)^3}$ *A1*

(c) $\boxed{1.3457}$ *B1*

STRETCH

E3 **(a)** $y' = e^{-x}(2\cos 2x - \sin 2x)$ *M1*

$y'' = e^{-x}(-3\sin 2x - 4\cos 2x)$ *A1*

$= -2y' - 5y$, as required *A1**

(b) $y(0) = 0$, $y'(0) = 2$, $y''(0) = -4$ *B1*

$y''' = -2y'' - 5y'$ gives -2; $y'''' = -2y''' - 5y''$ gives 24 *M1 A1 A1*

$\boxed{2x - 2x^2 - \frac{1}{3}x^3 + x^4}$ *A1*

(c) $\boxed{-2}$ *B1*