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INTERNATIONAL A LEVEL MATHEMATICS • FURTHER PURE 2 (WFM02)

Further Pure 2 Practice Paper 3

Worked Solutions

AIMING FOR A/A*

Written in the style of the Pearson Edexcel International Advanced Level paper (WFM02/01) • 75 marks

Every question solved in full on the paper itself, the way your teacher would write it out

How to read these solutions. Each page is the real question page, with a full answer written on the answer lines.

Red pen

M1 A1

boxed

the working a full-marks answer shows, with a short note on why each step is taken

the mark earned on that line: M = method, A = accuracy, B = independent mark

the final answer to each part

Sit the paper first. These solutions are most useful after you have tried every question yourself under timed conditions. Mark your paper with the mark scheme, then use this booklet to see exactly how each lost mark could have been earned.

Q	Topic	Marks	Page
1	A modulus against a rational function	10	2
2	A sum of logarithms and the least n	10	3
3	de Moivre's theorem and exact roots	11	4
4	The image of a quarter disc	11	5
5	A second-order equation by substitution	11	6
6	A product of series and a limit	11	7
7	Tangents and the area of a limaçon	11	8

Question paper & mark scheme | Exam questions by topic | More practice papers

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1. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

(a) On the same axes, sketch the curve with equation $y = \dfrac{x+3}{x-1}$ and the graph of $y = |x-3|$, stating the equations of any asymptotes.

(3)

(b) Use algebra to find the set of values of x for which

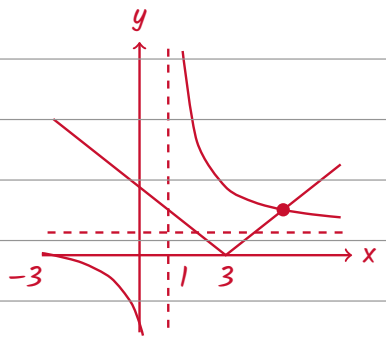
$$\frac{x+3}{x-1} \geqslant |x-3|$$

(7)

(a) $y = \frac{x+3}{x-1}$: asymptotes $x = 1$, $y = 1$; crosses the axes at $(-3, 0)$ and $(0, -3)$ B1

$y = |x - 3|$: V shape, vertex $(3, 0)$, through $(0, 3)$ B1

B1



(b) Case $x \geqslant 3$: $|x - 3| = x - 3$ and $x - 1 > 0$: $x + 3 = (x - 3)(x - 1)$ at the crossing M1

$x + 3 = x^2 - 4x + 3 \Rightarrow x^2 - 5x = 0 \Rightarrow x = 0$ or $x = 5$ A1

$x = 0$ is not in the case $x \geqslant 3$, so reject it: the only crossing here is $x = 5$ B1

Case $x < 3$: $|x - 3| = 3 - x$: $x + 3 = (3 - x)(x - 1) \Rightarrow x^2 - 3x + 6 = 0$ M1

Discriminant $9 - 24 = -15 < 0$: no real roots, so the graphs do not meet for $x < 3$ A1

Critical values $x = 1$ (asymptote) and $x = 5$; the curve is on or above the V between them dM1

$1 < x \leqslant 5$ ($x = 1$ excluded: not defined; $x = 5$ included: equality) A1

(Total for Question 1 is 10 marks)

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2. **(a)** Show that, for $r \geqslant 1$,

$$\ln \frac{(r+1)^2}{r(r+2)} = \ln \frac{r+1}{r+2} - \ln \frac{r}{r+1}$$

(2)

(b) Hence, using the method of differences, find $\sum_{r=1}^{n} \ln \frac{(r+1)^2}{r(r+2)}$, giving your answer as a single logarithm in terms of n

(4)

(c) Find the least value of n for which

$$\sum_{r=1}^{n} \ln \frac{(r+1)^2}{r(r+2)} > \ln 1.99$$

(4)

(a) $\ln \frac{r+1}{r+2} - \ln \frac{r}{r+1} = \ln\left(\frac{r+1}{r+2} \times \frac{r+1}{r}\right)$ (subtracting logs = dividing) M1

$= \ln \frac{(r+1)^2}{r(r+2)}$ as required A1

(b) $r = 1$: $\ln \frac{2}{3} - \ln \frac{1}{2}$ $r = 2$: $\ln \frac{3}{4} - \ln \frac{2}{3}$ $r = 3$: $\ln \frac{4}{5} - \ln \frac{3}{4}$ M1

... $r = n$: $\ln \frac{n+1}{n+2} - \ln \frac{n}{n+1}$

Everything cancels except $\ln \frac{n+1}{n+2} - \ln \frac{1}{2}$ A1

$= \ln\left(\frac{n+1}{n+2} \div \frac{1}{2}\right)$ M1

$\ln(2(n+1)/(n+2))$ A1

(c) ln is increasing, so we need $\frac{2(n+1)}{n+2} > 1.99$ M1

$2n + 2 > 1.99n + 3.98 \Rightarrow 0.01n > 1.98$ M1

$n > 198$ ($n = 198$ gives exactly 1.99, which is not > 1.99) A1

$n = 199$ A1

(Total for Question 2 is 10 marks)

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3. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

(a) Use de Moivre's theorem to show that

$$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$$

(5)

(b) Hence find the four roots of the equation

$$16x^4 - 20x^2 + 5 = 0$$

giving each root in the form $\sin k\pi$, where $-\frac{1}{2} < k < \frac{1}{2}$

(4)

(c) Hence find the exact value of $\sin\frac{\pi}{5}\sin\frac{2\pi}{5}$

(2)

(a) $c = \cos\theta$, $s = \sin\theta$: $\cos 5\theta + i\sin 5\theta = (c + is)^5$, so $\sin 5\theta = \text{Im}(c + is)^5$ B1

Imaginary terms: $5c^4(is) + 10c^2(is)^3 + (is)^5 = i(5c^4s - 10c^2s^3 + s^5)$ M1

$\sin 5\theta = 5c^4s - 10c^2s^3 + s^5$ A1

Use $c^2 = 1 - s^2$: $5(1 - s^2)^2s - 10(1 - s^2)s^3 + s^5 = 5s - 10s^3 + 5s^5 - 10s^3 + 10s^5 + s^5$ M1

$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$ as required A1

(b) $\sin 5\theta = \sin\theta(16\sin^4\theta - 20\sin^2\theta + 5)$: $\sin 5\theta = 0$, $\sin\theta \neq 0$ gives a root M1

$5\theta = k\pi$: $\theta = \pm\frac{\pi}{5}, \pm\frac{2\pi}{5}$ ($\theta = 0$ gives $\sin\theta = 0$, so not a root) M1

$\sin\frac{3\pi}{5} = \sin\frac{2\pi}{5}$ and $\sin\frac{4\pi}{5} = \sin\frac{\pi}{5}$ are repeats, so the four roots are: A1

$\boxed{\sin\frac{\pi}{5},\ \sin\left(-\frac{\pi}{5}\right),\ \sin\frac{2\pi}{5},\ \sin\left(-\frac{2\pi}{5}\right)}$ A1

(c) Product of the roots $= \frac{5}{16}$: $\sin^2\frac{\pi}{5}\sin^2\frac{2\pi}{5} = \frac{5}{16}$ M1

Both sines are positive, so $\sin\frac{\pi}{5}\sin\frac{2\pi}{5} = \boxed{\sqrt{5}/4}$ A1

(Total for Question 3 is 11 marks)

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4. The transformation T from the z-plane to the w-plane, where $w = u + iv$, is given by

$$w = \frac{z-1}{z+1}, \qquad z \neq -1$$

(a) Show that T maps the circle $|z| = 1$, $z \neq -1$, onto the imaginary axis in the w-plane.

(3)

The region R of the z-plane is given by $|z| < 1$ and $0 < \arg z < \frac{\pi}{2}$

(b) Find the image of R under T, and shade it on an Argand diagram.

(5)

(c) Find the set of possible values of $|w-1|$ for points w in the image of R

(3)

(a) $w(z+1) = z - 1 \Rightarrow z = \frac{1+w}{1-w}$. $|z| = 1$: $|1+w| = |1-w|$ M1

$(1+u)^2 + v^2 = (1-u)^2 + v^2 \Rightarrow 4u = 0$ M1

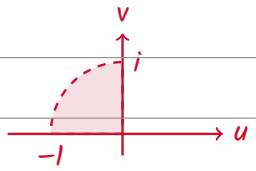
$u = 0$, which is the imaginary axis as required A1

(b) $z = \frac{(1+w)(1-w^*)}{|1-w|^2} = \frac{1 - w^* + w - |w|^2}{|1-w|^2} = \frac{1-|w|^2 + 2iv}{|1-w|^2}$ M1

$0 < \arg z < \frac{\pi}{2}$ means $\text{Re}\, z > 0$ and $\text{Im}\, z > 0$. $\text{Re}\, z > 0$: $1 - |w|^2 > 0 \Rightarrow |w| < 1$ A1

$\text{Im}\, z > 0$: $v > 0$. $|z| < 1$: $|1+w| < |1-w| \Rightarrow u < 0$ A1

Image: $|w| < 1$, $u < 0$, $v > 0$ (the open quarter disc in the second quadrant) A1



B1

(c) $|w-1|$ = distance from 1: nearest point 0 (distance 1), furthest -1 (distance 2) M1

0 and -1 are on the boundary, not in the region (the region is open) A1

$1 < |w-1| < 2$ A1

(Total for Question 4 is 11 marks)

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5.

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = x^3, \qquad x > 0$$

(a) Show that the substitution $x = e^u$ transforms this differential equation into

$$\frac{d^2y}{du^2} - 3\frac{dy}{du} + 2y = e^{3u}$$

(5)

(b) Hence find the general solution of the original differential equation, giving y in terms of x

(4)

(c) Find the particular solution for which $y = 0$ and $\frac{dy}{dx} = 1$ when $x = 1$

(2)

(a) $x = e^u$, so $u = \ln x$ and $\frac{du}{dx} = \frac{1}{x}$: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{x}\frac{dy}{du}$ B1

$\frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{du} + \frac{1}{x} \times \frac{d^2y}{du^2} \times \frac{du}{dx}$ (product rule, then chain rule) M1

$= \frac{1}{x^2}\left(\frac{d^2y}{du^2} - \frac{dy}{du}\right)$ A1

Substitute: $\left(\frac{d^2y}{du^2} - \frac{dy}{du}\right) - 2\frac{dy}{du} + 2y = (e^u)^3$ M1

$\frac{d^2y}{du^2} - 3\frac{dy}{du} + 2y = e^{3u}$ as required A1

(b) $m^2 - 3m + 2 = 0 \Rightarrow (m-1)(m-2) = 0$: $CF = Ae^u + Be^{2u}$ M1

PI $y = \lambda e^{3u}$: $9\lambda - 9\lambda + 2\lambda = 1 \Rightarrow \lambda = \frac{1}{2}$ M1

$y = Ae^u + Be^{2u} + \frac{1}{2}e^{3u}$, and $e^u = x$: A1

$\boxed{y = Ax + Bx^2 + \frac{1}{2}x^3}$ A1

(c) $y(1) = 0$: $A + B + \frac{1}{2} = 0$; $y' = A + 2Bx + \frac{3}{2}x^2$, $y'(1) = 1$: $A + 2B = -\frac{1}{2}$; $B = 0$, $A = -\frac{1}{2}$ M1

$\boxed{y = \frac{1}{2}x^3 - \frac{1}{2}x}$ A1

(Total for Question 5 is 11 marks)

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6. **(a)** Using the Maclaurin series for $\ln(1+x)$, find the series expansion of $\ln(1+2x)$ in ascending powers of x, up to and including the term in x^3, and state the range of values of x for which the expansion is valid.

(4)

(b) Hence find the series expansion of $e^x \ln(1+2x)$ in ascending powers of x, up to and including the term in x^3

(4)

(c) Hence find

$$\lim_{x \to 0} \frac{e^x \ln(1+2x) - 2x}{x^3}$$

(2)

(d) Explain why the series in part (b) should not be used to estimate the value of $e \ln 3$

(1)

(a) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$: replace x by $2x$ M1

$\ln(1+2x) = 2x - \frac{4x^2}{2} + \frac{8x^3}{3} - \ldots = 2x - 2x^2 + \ldots$ A1

$\boxed{\ln(1+2x) \approx 2x - 2x^2 + \frac{8}{3}x^3}$ A1

Valid for $-1 < 2x \leqslant 1$: $\boxed{-\frac{1}{2} < x \leqslant \frac{1}{2}}$ B1

(b) $e^x = 1 + x + \frac{x^2}{2} + \ldots$ B1

$(1 + x + \frac{1}{2}x^2)(2x - 2x^2 + \frac{8}{3}x^3)$: x: 2; x^2: $-2 + 2$; x^3: $\frac{8}{3} - 2 + 1$ M1

x^2 coefficient $= 0$ A1

$\boxed{e^x \ln(1+2x) \approx 2x + \frac{5}{3}x^3}$ A1

(c) Top $= 2x + \frac{5}{3}x^3 + \ldots - 2x = \frac{5}{3}x^3 +$ (higher powers); divide by x^3: $\frac{5}{3} +$ (terms in x) M1

As $x \to 0$ the terms in $x \to 0$: limit $= \boxed{5/3}$ A1

(d) $e \ln 3$ needs $x = 1$, outside $-\frac{1}{2} < x \leqslant \frac{1}{2}$, so the series for $\ln(1+2x)$ does not converge B1

(Total for Question 6 is 11 marks)

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7.

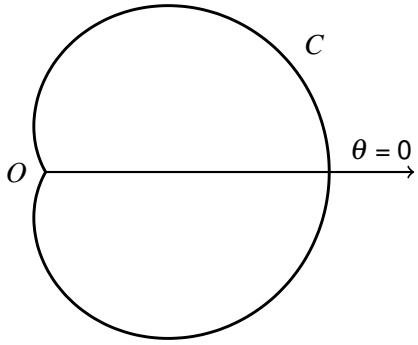

Figure 1

Figure 1 shows the curve C with polar equation

$$r = 1 + 2\cos\theta, \qquad -\frac{2\pi}{3} \leqslant \theta \leqslant \frac{2\pi}{3}$$

(a) Find the polar coordinates of each point on C at which the tangent to C is perpendicular to the initial line, giving any non-exact values of θ to 3 significant figures.

(6)

(b) Use calculus to find the exact area of the region enclosed by C

(5)

(a) Perpendicular to the initial line means $\frac{dx}{d\theta} = 0$, where $x = r\cos\theta = \cos\theta + 2\cos^2\theta$ M1

$\frac{dx}{d\theta} = -\sin\theta - 4\cos\theta\sin\theta$ M1

$= -\sin\theta(1 + 4\cos\theta) = 0$ A1

$\sin\theta = 0 \Rightarrow \theta = 0,$ or $\cos\theta = -\frac{1}{4} \Rightarrow \theta = \pm 1.823\ldots$ (inside $|\theta| \leqslant 2.09$) M1

$\theta = 0$: $r = 3$ $(3, 0)$ A1

$\cos\theta = -\frac{1}{4}$: $r = 1 - \frac{1}{2} = \frac{1}{2}$ $(\frac{1}{2}, 1.82)$ and $(\frac{1}{2}, -1.82)$ A1

(b) Area $= \frac{1}{2}\int (1 + 2\cos\theta)^2\, d\theta = \frac{1}{2}\int (1 + 4\cos\theta + 4\cos^2\theta)\, d\theta$ M1

$4\cos^2\theta = 2 + 2\cos 2\theta$: $= \frac{1}{2}\int (3 + 4\cos\theta + 2\cos 2\theta)\, d\theta$ M1

$= \frac{1}{2}[3\theta + 4\sin\theta + \sin 2\theta]_{-2\pi/3}^{2\pi/3}$ A1

$= [3\theta + 4\sin\theta + \sin 2\theta]_0^{2\pi/3} = 2\pi + 2\sqrt{3} - \frac{\sqrt{3}}{2}$ ($r \geqslant 0$ only on this interval) M1

$2\pi + 3\sqrt{3}/2$ A1

(Total for Question 7 is 11 marks)

TOTAL FOR PAPER IS 75 MARKS

END

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