



eClassroom

A-LEVEL MATHEMATICS • PURE 4

Proof

Worked Solutions

Pearson Edexcel International A Level (WMA14) & A Level (9MA0) • Calculator allowed: show every step

Full worked solutions for the free Proof worksheet, A-Level and International A-Level Maths



Tap a section to jump straight to it

Full worked solutions for every question on the Pure 4 Proof worksheet, in the same order and with the same question numbers.

How to mark your work. Tick each step you got right, not just the final answer. At A-level most marks are for method, so a wrong final answer can still earn most of the marks. If you lost marks, find the *first* line where your working is different from ours. That is the thing to fix.

In Section E: **M** = method mark, **dM** = method mark that depends on the previous M mark, **A** = accuracy mark (needs the M mark before it), **B** = independent mark.

Question sheet

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A

Your turn answers

Compare each line of your working with the solution.

CORE

A1 Assume $n^2 + 4n$ is odd and n is even: $n = 2k$

$$n^2 + 4n = 4k^2 + 8k = 2(2k^2 + 4k), \text{ which is even: a contradiction}$$

so if $n^2 + 4n$ is odd, then n is odd

A2 Assume $\sqrt{5} = \frac{a}{b}$, where a and b are integers with no common factors. Then $a^2 = 5b^2$

$$a^2 \text{ is a multiple of } 5, \text{ so } a \text{ is a multiple of } 5: a = 5m, \text{ so } 25m^2 = 5b^2 \text{ and } b^2 = 5m^2$$

$$b^2 \text{ is a multiple of } 5, \text{ so } b \text{ is too: } \mathbf{5 \text{ is a common factor, a contradiction, so } \sqrt{5} \text{ is irrational}}$$

A3 Assume they meet: $x^4 + 3 = 2x^2 + 1$, so $x^4 - 2x^2 + 2 = 0$

$$(x^2 - 1)^2 + 1 = 0, \text{ so } (x^2 - 1)^2 = -1: \mathbf{a \text{ square cannot be negative, a contradiction, so they do not meet}}$$

STRETCH

A4 Assume $2, 3, \dots, p_n$ are all the primes. Then $N \geq 2 \times 3 - 1 = 5$, so $N > 1$ and N has a prime factor q is one of the primes in the list, so q divides $2 \times 3 \times \dots \times p_n$ and N , so it divides their difference, 1

no prime divides 1: a contradiction, so there are infinitely many primes

A5 Assume $\sqrt[3]{4} = \frac{a}{b}$ (no common factors): $a^3 = 4b^3$ is even, so a is even: $a = 2m$

$$8m^3 = 4b^3 \Rightarrow b^3 = 2m^3, \text{ so } b^3 \text{ is even and } b \text{ is even}$$

2 is a common factor: a contradiction, so $\sqrt[3]{4}$ is irrational

A6 Assume $e^x + 4e^{-x} < 4$ for some x . $e^x > 0$, so multiplying by it keeps the inequality: $e^{2x} - 4e^x + 4 < 0$

$$(e^x - 2)^2 < 0: \mathbf{a \text{ square cannot be negative: a contradiction}}$$

A7 Assume $\frac{dy}{dx} = 0$ for some x . $\frac{dy}{dx} = 3x^2 - 6x + 5 = 3(x - 1)^2 + 2 \geq 2$

$\frac{dy}{dx}$ is never 0: a contradiction, so there are no stationary points



B

Fluency solutions

One mark for each correct part. The notes in *italics* explain why each sequence was chosen.

CORE

B1 (a) **there is an integer n for which $n^2 + n$ is odd**

(b) **there are integers a and b such that $3a + 6b = 1$**

(c) **$\sqrt{11}$ is rational: $\sqrt{11} = \frac{a}{b}$ with no common factors**

(d) **there is an integer n with n^2 odd and n even**

"For all" becomes "there is one"; "there are no" becomes "there are"; an "if" keeps its condition and denies its conclusion.

B2 (a) n even: $14k + 2 = 2(7k + 1)$ is even: **contradiction** (b) n even: $8k^3 = 2(4k^3)$ is even: **contradiction**

(c) n odd: $(2k + 1)^2 + 2(2k + 1) = 2(2k^2 + 4k + 1) + 1$ is odd: **contradiction**

(d) n odd: $5(2k + 1)^2 - 1 = 2(10k^2 + 10k + 2)$ is even: **contradiction**

Each time, assume the condition holds and the conclusion fails, then reach the opposite of the condition.

B3 (a) $3(2a + 3b) = 2$, so $2a + 3b = \frac{2}{3}$, not an integer: **contradiction**

(b) $5(2a - 3b) = 4$, so $2a - 3b = \frac{4}{5}$: **contradiction**

(c) $2(2a + 3b)$ is even but 9 is odd: **contradiction**

(d) $6(2a - 3b + 5c) = 8$, so $2a - 3b + 5c = \frac{4}{3}$: **contradiction**

The common factor of the coefficients must divide the right-hand side.

B4 Assume each number equals a rational r

(a) $\sqrt{2} = \frac{r}{3}$ (b) $\sqrt{2} = r - 5$ (c) $\sqrt{2} = \frac{1}{r}$ (d) $3 + 2\sqrt{2} = r$, so $\sqrt{2} = \frac{r-3}{2}$

each right-hand side is rational, but $\sqrt{2}$ is irrational: a contradiction

Rearrange to make $\sqrt{2}$ the subject; rationals stay rational under $+$, $-$, $\times$ and $\div$.

B5 Assume they meet, then eliminate y :

(a) $x^2 + 2x + 6 = 0$: $(x + 1)^2 + 5 = 0$ (b) $x^2 - 4x + 5 = 0$: $(x - 2)^2 + 1 = 0$

(c) $x^4 - 4x^2 + 5 = 0$: $(x^2 - 2)^2 + 1 = 0$ each needs a negative square: a contradiction

The same finish each time; in (c) the square is of $x^2 - 2$, not $x - 2$



B6 Assume the inequality fails for some value.

(a) $x^2 - 6x + 9 < 0$: $(x-3)^2 < 0$ (b) $\times x > 0$: $4x^2 - 4x + 1 < 0$: $(2x-1)^2 < 0$

(c) $a^2 - 2ab + b^2 < 0$: $(a-b)^2 < 0$ (d) $\times(x-3) > 0$: $x^2 - 10x + 25 < 0$: $(x-5)^2 < 0$

Multiply only by something you know is positive, and say so. Every part ends with an impossible square.

STRETCH

B7 Assume each is $\frac{a}{b}$ with no common factors.

(a) $a^3 = 5b^3$: $5 \mid a$, $a = 5m$, $b^3 = 25m^3$, $5 \mid b$: **contradiction**

(b) $a^3 = 9b^3$: $3 \mid a$, $a = 3m$, $b^3 = 3m^3$, $3 \mid b$: **contradiction**

(c) $a^4 = 8b^4$: $a = 2m$, $16m^4 = 8b^4$, $b^4 = 2m^4$, b even: **contradiction**

(d) $a^3 = 6b^3$: $a = 2m$, $4m^3 = 3b^3$, b even: **contradiction**

Use a prime factor of the number under the root. In (d), $3b^3$ is even and 3 is odd, so b^3 is even.

B8 (a) $a^2 = 4b + 2$ is even, so a is even and a^2 is a multiple of 4; $4b + 2$ is not: **contradiction**

(b) $a^2 = 4b + 3$ is odd, so $a = 2k + 1$ and $a^2 = 4(k^2 + k) + 1$, never $4b + 3$: **contradiction**

(c) every square is $4k$ or $4k + 1$, so $a^2 + b^2$ is $4k$, $4k + 1$ or $4k + 2$: **never $4c + 3$, contradiction**

(d) a^3 is even, so $a = 2m$ and $a^3 = 8m^3$, a multiple of 4; $4b + 2$ is not: **contradiction**

Find the possible remainders on division by 4, then show the right-hand side is not one of them.

B9 Assume a stationary point (or a meeting point) exists.

(a) $\frac{dy}{dx} = 3x^2 - 12x + 15 = 3(x-2)^2 + 3 \geq 3$: **never 0**

(b) $\frac{dy}{dx} = 5 + 3\cos x + 2\sin x$, and $3\cos x + 2\sin x = \sqrt{13}\cos(x - \alpha) \geq -\sqrt{13}$: $\geq 5 - \sqrt{13} > 0$

(c) $2 + \frac{2x}{x^2 + 1} = 0 \Rightarrow x^2 + x + 1 = 0$, discriminant $1 - 4 = -3 < 0$: **no real x**

(d) $e^{2x} - 2e^x + 3 = 0$: $(e^x - 1)^2 + 2 = 0$, **impossible**

Four different reasons for "never 0": a square, the range of $R\cos$, a discriminant, a square in e^x



B10 Assume each inequality fails for some value.

(a) $x^4 + 1 > 0$, so $2x^2 > x^4 + 1$: $(x^2 - 1)^2 < 0$

(b) $\sin^4 x + \cos^4 x = 1 - 2\sin^2 x \cos^2 x = 1 - \frac{1}{2}\sin^2 2x$; so $\sin^2 2x > 1$: **impossible**

(c) $u = \sqrt{x^2 + 4} \geq 2$: $\frac{u^2 + 1}{u} < \frac{5}{2}$, so $(2u - 1)(u - 2) < 0$: $\frac{1}{2} < u < 2$, but $u \geq 2$

(d) $\times ab > 0$: $(a + b)^2 < 4ab$: $(a - b)^2 < 0$ each gives a contradiction

Each needs a first move: clear a positive denominator, use an identity, substitute and keep the range of the new letter.

C

Reasoning solutions

Your wording can be different. Check that your reason makes the same point.

CORE

C1 The opposite of “for all n ” is “for some n ”, not “for all n ”.

Assume there is an integer n for which $n^2 - n$ is odd.

C2 “Solutions relying on calculator technology” score no marks, and she shows no algebra.

$x^4 - 3x^2 + 5 = \left(x^2 - \frac{3}{2}\right)^2 + \frac{11}{4} > 0$, so it is never 0: a contradiction

C3 Examples cannot cover every integer. Assume n^2 is a multiple of 3 but n is not: $n = 3k + 1$ or $3k + 2$
 $(3k + 1)^2 = 3(3k^2 + 2k) + 1$ and $(3k + 2)^2 = 3(3k^2 + 4k + 1) + 1$, neither a multiple of 3

a contradiction, so n is a multiple of 3

C4 $x = 0$ is not allowed: $\frac{1}{0}$ is undefined, so the statement says nothing about it.

$x = -1$: $-1 + \frac{1}{-1} = -2 < 2$. The step “multiply by x ” keeps the inequality only because $x > 0$

STRETCH

C5 “ a^2 is a multiple of 9, so a is a multiple of 9” is false: **$a = 3$: $a^2 = 9$, but 3 is not a multiple of 9**

The step works for a prime such as 5, but 9 is not prime. Indeed $\sqrt{9} = 3$ is rational.



C6 $x^2 > 0$ is false when $x = 0$; it should be $x^2 \geq 0$ and $x^4 \geq 0$

then $x^4 + 5x^2 + 4 \geq 4 > 0$, so it still cannot be 0, and the conclusion still follows.

C7 $2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1 = 30031 = 59 \times 509$, so it is not prime.

The proof only needs a prime factor of N that is not in the list. 59 and 509 are new primes, as the proof says.

C8 (a) $n = 3k$: $9k^2 = 3(3k^2)$; $n = 3k \pm 1$: $9k^2 \pm 6k + 1 = 3(3k^2 \pm 2k) + 1$: **remainder 0 or 1**

(b) Assume $a^2 + b^2$ is a multiple of 3 but a and b are not both multiples of 3. One square has remainder 1 and the other 0 or 1

so $a^2 + b^2$ has remainder 1 or 2: **not a multiple of 3, a contradiction**

D

Problem solving solutions

There is often more than one correct method. Any complete, correct method scores full marks.

CORE

D1 Assume $n^3 + 5$ is odd and n is odd: $n = 2k + 1$

$$n^3 + 5 = 8k^3 + 12k^2 + 6k + 6 = 2(4k^3 + 6k^2 + 3k + 3), \text{ which is even}$$

a contradiction, so if $n^3 + 5$ is odd, then n is even

D2 (a) Assume $\sqrt{3} = \frac{a}{b}$, integers with no common factors: $a^2 = 3b^2$, so $a = 3m$ and $b^2 = 3m^2$, so $3 \mid b$

3 is a common factor: a contradiction, so $\sqrt{3}$ is irrational

(b) $(2 + \sqrt{3})^2 = 7 + 4\sqrt{3}$. If it equalled a rational r , then $\sqrt{3} = \frac{r-7}{4}$ would be rational: **contradiction**

D3 Assume C and l meet: $x^2 + 2x + 7 = 4x + 5$, so $x^2 - 2x + 2 = 0$

$(x-1)^2 + 1 = 0$, so $(x-1)^2 = -1$: **impossible, so C and l do not meet**

D4 The other side is $\frac{36}{x}$, so the perimeter is $2x + \frac{72}{x}$. Assume it is less than 24 for some $x > 0$

$x > 0$, so $2x^2 + 72 < 24x$, so $2(x-6)^2 < 0$: **impossible, so the perimeter is at least 24 cm**



STRETCH

D5 (a) $\frac{dy}{dx} = 3x^2 - 6x + 4 - \sin x = 3(x-1)^2 + 1 - \sin x$ **as required**

(b) Assume $\frac{dy}{dx} = 0$ for some x . $3(x-1)^2 \geq 0$ and $1 - \sin x \geq 0$, so both must be 0

$x = 1$ and $\sin x = 1$; but $\sin x = 1$ only at $x = \frac{\pi}{2} + 2n\pi$, never at $x = 1$: **a contradiction**

D6 Assume $\frac{a}{b}$ is a solution, with no common factors, so a and b are not both even. Then $a^3 + ab^2 + b^3 = 0$

both odd: odd + odd + odd is odd; a even, b odd: even + even + odd; a odd, b even: odd + even + even

the left-hand side is always odd, so it cannot be 0: a contradiction

D7 (a) $\sin^2 x + 2 \sin x \cos x + \cos^2 x = 1 + \sin 2x$ **as required**

(b) Assume $\sin x + \cos x \geq -1$ for some x in the interval. Both are negative there, so $-1 \leq \sin x + \cos x < 0$

So $(\sin x + \cos x)^2 \leq 1$, giving $\sin 2x \leq 0$; but $360^\circ < 2x < 540^\circ$, so $\sin 2x > 0$: **contradiction**

D8 (a) $(4j+1)(4k+1) = 16jk + 4j + 4k + 1 = 4(4jk + j + k) + 1$ **of the form $4k+1$**

(b) $N = 4(p_1 \cdots p_r - 1) + 3$ is odd, so its prime factors are of the form $4k+1$ or $4k+3$

If all were $4k+1$, N would be $4k+1$ by (a). So a prime factor q is $4k+3$, so q is one of the p_i

q divides $4p_1 \cdots p_r$ and N , so it divides 1: **a contradiction**

E

Exam practice solutions

Mark-scheme notes are shown on the right of each line.

CORE

Exam questions for everyone

E1 (a) $(x^2 - 2x)^2 = x^4 - 4x^3 + 4x^2$ and $(2x - 1)^2 = 4x^2 - 4x + 1$

M1: both squares expanded

adding 2: $x^4 - 4x^3 + 8x^2 - 4x + 3$ **as required**

A1*

(b) Assume C_1 and C_2 meet: $x^4 - 2x^3 + 3x^2 = 2x^3 - 5x^2 + 4x - 3$ for some x

B1: assumption

so $x^4 - 4x^3 + 8x^2 - 4x + 3 = 0$, i.e. $(x^2 - 2x)^2 + (2x - 1)^2 + 2 = 0$

M1: uses (a)

the left side is at least 2: **a contradiction, so they do not intersect**

A1: reason and conclusion



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- E2** (a) Assume n^3 is even and n is odd: $n = 2k + 1$ B1
 $n^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1$, odd: **contradiction** B1
- (b) Assume $\sqrt[3]{12} = \frac{a}{b}$, where a and b are integers with no common factors B1
 $a^3 = 12b^3 = 2(6b^3)$ is even, so a is even by (a): $a = 2m$ M1
 $8m^3 = 12b^3 \Rightarrow 2m^3 = 3b^3$, so $3b^3$ is even; 3 is odd, so b^3 is even and b is even M1
2 is a common factor: a contradiction, so $\sqrt[3]{12}$ is irrational A1*

STRETCH

- E3** (a) $S = ka + (0 + 1 + \dots + (k - 1)) = ka + \frac{(k - 1)k}{2}$ M1
 $S = \frac{2ka + k^2 - k}{2} = \frac{k(2a + k - 1)}{2}$ **as required** A1*
- (b) Their sum is $2a + 2k - 1$, which is odd, so **one is odd and the other even** B1
- (c) Assume $2^m = \frac{k(2a + k - 1)}{2}$ for some $k \geq 2$, $a \geq 1$: $k(2a + k - 1) = 2^{m+1}$ B1
 $k \geq 2$ and $2a + k - 1 \geq 3$, so both factors are greater than 1 M1
by (b) one is odd, so 2^{m+1} has an odd factor greater than 1 M1
but every factor of 2^{m+1} greater than 1 is even: a contradiction A1
- (d) **$40 = 6 + 7 + 8 + 9 + 10$** B1

