



eClassroom

A-LEVEL MATHEMATICS • PURE 2

Differentiation

Worksheet

Pearson Edexcel International A Level (WMA12) & A Level (9MA0) • Calculator allowed: show every step

Free Differentiation worksheet for A-Level and International A-Level Maths, with full worked solutions



Tap a section to jump straight to it

By the end of this worksheet you can: stationary points and their nature • increasing and decreasing • *Stretch:* optimisation and proofs.

Which questions are for me?

CORE The methods every A-level student must secure. Start here.

STRETCH Optimising shapes, stationary points with fractional powers, and a test that fails.

Every section has both, clearly labelled. Questions get harder as you go, and every skill in Sections A to D is needed in Section E.

Track your progress

Section	B	C	D	E
Core marks	23	10	13	11
Stretch marks	16	14	17	8
My score				

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A

Worked examples & Your turn

Read each example carefully. Cover it up, then answer the question beside it using the same method.

CORE

Secure the method

Worked example A1

Find the stationary points of $y = x^3 - 3x^2 - 24x + 5$

Solve $\frac{dy}{dx} = 0$, then put each x into y

$$3x^2 - 6x - 24 = 3(x - 4)(x + 2) = 0$$

$(4, -75), (-2, 33)$

Your turn A1

Find the stationary points of $y = 2x^3 + 3x^2 - 36x + 1$

Worked example A2

Find the stationary point of $y = x^2 + \frac{16}{x}$, $x > 0$, and determine its nature.

$$2x - 16x^{-2} = 0, \text{ so } x^3 = 8, x = 2, y = 12$$

$$\frac{d^2y}{dx^2} = 2 + 32x^{-3} = 6 > 0$$

$(2, 12)$ is a minimum

Your turn A2

Find the stationary point of $y = x + \frac{9}{x}$, $x > 0$, and determine its nature.

Worked example A3

Find the set of values of x for which $f(x) = x^3 - 12x$ is decreasing.

Decreasing where $f'(x) < 0$: $3x^2 - 12 < 0$

$$3(x - 2)(x + 2) < 0: \text{ between the roots}$$

$-2 < x < 2$

Your turn A3

Find the set of values of x for which $f(x) = 2x^3 - 9x^2 + 7$ is increasing.



STRETCH

Worked example A4: increasing for all x

Show that $f(x) = x^3 - 3x^2 + 4x$ is increasing for all x

$$f'(x) = 3x^2 - 6x + 4 = 3(x-1)^2 + 1$$

$$3(x-1)^2 \geq 0, \text{ so } f'(x) \geq 1$$

$$f'(x) > 0 \text{ for all } x$$

Your turn A4

Show that $f(x) = x^3 + 6x^2 + 15x$ is increasing for all x

Worked example A5: fractional powers

Find the stationary point of $y = x^{\frac{3}{2}} - 3x$, $x > 0$, and its nature.

$$\frac{3}{2}x^{\frac{1}{2}} = 3, \text{ so } x^{\frac{1}{2}} = 2, x = 4, y = -4$$

$$\frac{d^2y}{dx^2} = \frac{3}{4}x^{-\frac{1}{2}} > 0$$

$$(4, -4) \text{ is a minimum}$$

Your turn A5

Find the stationary point of $y = 6\sqrt{x} - x$, $x > 0$, and its nature.

Worked example A6: optimisation

A rectangle has perimeter 40 cm. Find its greatest possible area.

One variable: width x , length $20 - x$, so $A = 20x - x^2$

$$20 - 2x = 0, x = 10; \frac{d^2A}{dx^2} = -2 < 0$$

$$100 \text{ cm}^2$$

Your turn A6

A rectangle has area 64 cm^2 . Find its least possible perimeter.

Worked example A7: using stationary values

Find the values of k for which $x^3 - 3x = k$ has three real solutions.

Stationary points $(-1, 2)$ (maximum) and $(1, -2)$ (minimum). A horizontal line between them meets the curve three times.

$$-2 < k < 2$$

Your turn A7

Find the values of k for which $x^3 - 12x + 1 = k$ has three real solutions.



B**Fluency**

Each question is a designed sequence. From one part to the next, ask yourself: what changed, and what did it do to the answer?

CORE*Secure the method*

B1 Find the coordinates of the stationary points of

(a) $y = x^2 - 6x + 1$

(b) $y = 4x - x^2$

(c) $y = x^3 - 27x$

(d) $y = x^4 - 8x^2$

[4 marks]

B2 Use the second derivative to determine the nature of each stationary point you found in B1

(a) $y = x^2 - 6x + 1$

(b) $y = 4x - x^2$

(c) $y = x^3 - 27x$

(d) $y = x^4 - 8x^2$

[4 marks]

B3 Find the set of values of x for which

(a) $x^2 - 4x$ is increasing

(b) $6x - x^2$ is decreasing

(c) $x^3 - 3x^2$ is decreasing

(d) $x^3 + 3x$ is increasing

[4 marks]

B4 Find the stationary point of each curve, where $x > 0$

(a) $y = x + \frac{4}{x}$

(b) $y = x^2 + \frac{2}{x}$

(c) $y = \sqrt{x} - x$

(d) $y = x - 2\sqrt{x}$

[4 marks]



CORE

continued: optimisation and gradients

B5 Two positive numbers x and y satisfy $x + 2y = 20$, and their product is P

(a) Show that $P = 10x - \frac{1}{2}x^2$

(b) Find the value of x that makes P stationary.

(c) Find the greatest value of P , with a reason.

[3 marks]

B6 $f(x) = x^3 - 6x^2 + 9x$

(a) Find $f'(0)$ and say whether f is increasing or decreasing at $x = 0$

(b) Find $f'(2)$ and say whether f is increasing or decreasing at $x = 2$

(c) Find the x coordinates of the stationary points.

(d) Find the set of values of x for which f is decreasing.

[4 marks]

STRETCH

B7 Find the constants in each case.

(a) $y = x^3 + ax + 2$ has a stationary point where $x = 2$

(b) $y = ax^2 + bx$ has a stationary point at $(1, -3)$

(c) $y = x^2 + \frac{k}{x}$ has a stationary point where $x = 2$

(d) $y = kx^{\frac{3}{2}} - 6x$ has a stationary point where $x = 4$

[4 marks]

B8 By completing the square, show that

(a) $x^3 + 3x^2 + 6x$ is increasing for all x

(b) $x^3 - 6x^2 + 15x$ is increasing for all x

(c) $2x^3 + 3x^2 + 3x$ is increasing for all x

(d) $-x^3 + 3x^2 - 4x$ is decreasing for all x

[4 marks]

B9 Find the stationary value of each quantity for $x > 0$, and state whether it is a maximum or a minimum.

(a) $V = 12x^2 - x^3$

(b) $P = 40x - 2x^2$

(c) $C = 3x + \frac{75}{x}$

(d) $S = 2x^2 + \frac{32}{x}$

[4 marks]

B10 $f(x) = x^3 - 3x^2 + 4$ has stationary points $(0, 4)$ and $(2, 0)$. State the number of real solutions of $f(x) = k$ when

(a) $k = 2$

(b) $k = 4$

(c) $k = 5$

(d) $k = -1$

[4 marks]



C**Reasoning**

Show your working and explain your thinking. A correct answer without a reason does not score full marks.

CORE*Secure the method*

- C1** Sam finds the stationary point of $y = x^2 - 8x + 3$ as $(4, 0)$
Explain Sam's mistake, and give the correct stationary point. [2 marks]
- C2** Lily finds that $y = x^3 - 12x$ has a stationary point where $x = 2$, and writes "this is a minimum" with no working.
Explain why this would lose marks, and justify the nature of the point. [2 marks]
- C3** Kai finds the stationary points of $y = 5x^{\frac{3}{2}} - x^{\frac{5}{2}}, x \geq 0$. He divides $\frac{dy}{dx} = 0$ by $x^{\frac{1}{2}}$ and finds only $x = 3$
Explain what Kai has lost, and give all the stationary points. [3 marks]
- C4** $f'(x) = 3x^2 - 6x - 9$. Jo says f is decreasing for $x < -1$ or $x > 3$
Explain Jo's mistake, and give the correct set. [3 marks]



STRETCH

- C5** For $y = x^4$, Max finds $\frac{d^2y}{dx^2} = 0$ at the stationary point $(0, 0)$ and says it is “neither a maximum nor a minimum”.
Explain Max’s mistake, and find the nature of the point by another method. [3 marks]
- C6** Ali is asked for the greatest value of $f(x) = 12x - x^3$ for $0 \leq x \leq 3$. He works out $f(3) = 9$ and says this is the greatest value, because 3 is the largest x .
Explain Ali’s mistake, and find the greatest value. [3 marks]
- C7** $f(x) = x^3 - 3x$. Explain, using $f'(x)$, why the graph of $y = f(x)$ slopes downwards between $x = -1$ and $x = 1$, and describe the graph of $y = f'(x)$ [3 marks]
- C8** The curve $y = ax^2 + bx + c$, where a , b and c are constants and $a \neq 0$, has one stationary point.
- (a) Show that the stationary point is where $x = -\frac{b}{2a}$, and that it is a minimum when $a > 0$ (3)
- (b) Show that the stationary value is $c - \frac{b^2}{4a}$ (2)
- [5 marks]



D**Problem solving**

You decide the method. Read the whole question first, and show every step.

CORE

Secure the method

- D1** In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.

Figure 1 shows part of the curve C with equation

$$y = x^3 - 9x^2 + 24x - 10$$

The curve has a maximum point at A and a minimum point at B , as shown in Figure 1.

Use calculus to find the coordinates of A and of B , showing each step of your working. You do not need to prove which point is the maximum, but your answers must agree with Figure 1.

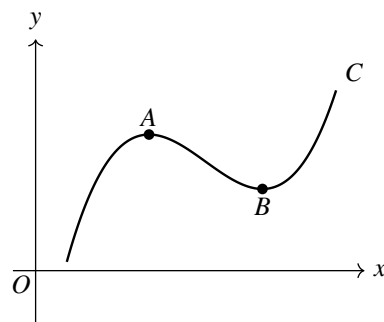


Figure 1

[3 marks]

- D2** In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.

Figure 2 shows the curve with equation

$y = 12\sqrt{x} - 2x$, $x \geq 0$. The curve has a stationary point P and meets the x -axis at the origin O and at the point Q .

(a) Using calculus, show that the only stationary point is at $x = 9$

(2)

(b) Use the second derivative to show that P is a maximum.

(1)

(c) Find the coordinates of Q .

(1)

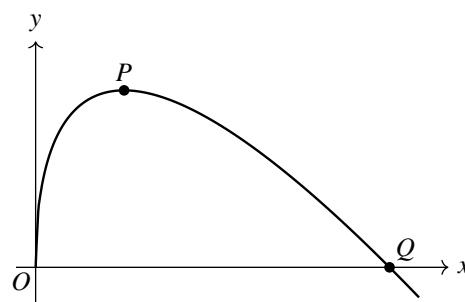


Figure 2

[4 marks]

- D3** $f(x) = 2x^3 - 3x^2 - 36x + 4$

The function f is defined for all real values of x . Using calculus, find the set of values of x for which $f(x)$ is an increasing function, giving your answer in set notation or as two inequalities, and showing clearly how you choose the regions from the roots of your quadratic.

[3 marks]



- D4** A farmer uses 40 m of fencing to make a rectangular pen against a long straight wall, as shown in Figure 3. The wall forms one side of the pen, so only three sides need fencing. The two sides at right angles to the wall each have length x metres.

(a) Show that the area of the pen is $A = 40x - 2x^2 \text{ m}^2$

(1)

(b) Use calculus to find the greatest possible area of the pen, and the length of fencing parallel to the wall, justifying that the area is a maximum.

(2)

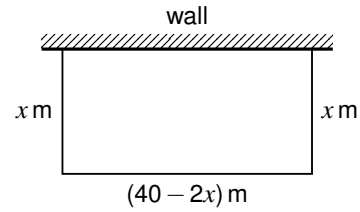


Diagram NOT accurately drawn

Figure 3

[3 marks]

STRETCH

- D5** Figure 4 shows an open box with a square base of side x cm and height h cm. The box has no lid. The volume of the box is 500 cm^3 , and its external surface area is $S \text{ cm}^2$

(a) Show that $S = x^2 + \frac{2000}{x}$

(2)

(b) Use calculus to find the least possible value of S , and the corresponding height of the box, justifying that S is a minimum.

(2)

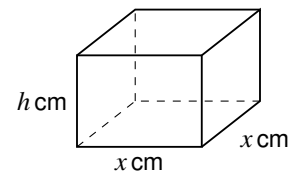


Diagram NOT accurately drawn

Figure 4

[4 marks]

- D6** Figure 5 shows the curve C with equation $y = 12 - x^2$ and a rectangle $PQRS$. The vertices S and R lie on the x -axis at $(-x, 0)$ and $(x, 0)$, and P and Q lie on C , where $0 < x < \sqrt{12}$

(a) Show that the area of the rectangle is $A = 24x - 2x^3$

(1)

(b) Use calculus to find the greatest possible area of the rectangle, and the coordinates of Q when this happens, justifying that it is a maximum.

(3)

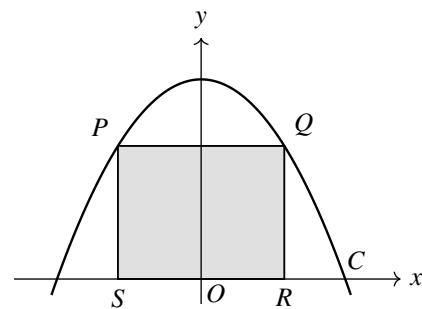


Figure 5

[4 marks]



- D7** In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.

After a storm, the water level in a river rises and then falls. The depth of water, D metres, in the river t hours after the storm is modelled by

$$D = \frac{1}{5}t^3 - 3t^2 + 12t + 10 \quad 0 \leq t \leq 8$$

- (a) Show that the depth is stationary when $t = 5 \pm \sqrt{5}$

(2)

- (b) Find the greatest depth of the river in this period, giving an exact answer, and justify that it is a maximum.

(2)

[4 marks]

- D8** Figure 6 shows the curve C with equation $y = x^4 - 4x^3$. The curve has a stationary point at the origin O and a minimum point at B .

- (a) Find the coordinates of the stationary points of C

(2)

- (b) Show that the second derivative is zero at O , and use the gradient on either side of O to show that O is neither a maximum nor a minimum.

(2)

- (c) Show that B is a minimum.

(1)

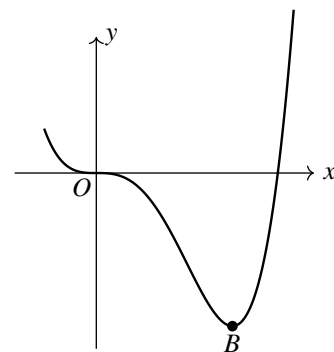


Figure 6

[5 marks]



E

Exam practice

Exam-style questions in the style of WMA12. Write your answers in the spaces, and show every step.

CORE

Exam questions for everyone

- E1** In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.

Figure 7 shows part of the curve C with equation

$$y = x^2 + \frac{54}{x} \quad x > 0$$

The curve has a minimum point P , as shown in Figure 7. The curve does not meet the x -axis or the y -axis.

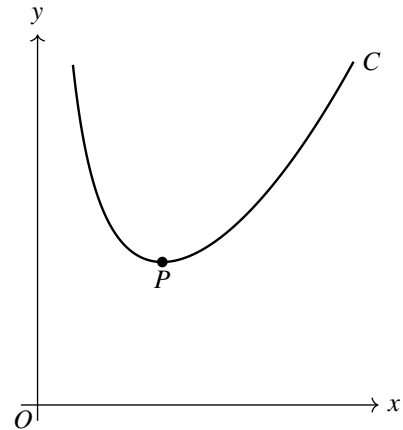


Figure 7

- (a) Find $\frac{dy}{dx}$

.....
(1)

- (b) Using calculus, find the coordinates of P

.....
(2)

- (c) Use the second derivative to show that P is a minimum point.

(1)

- (d) State the set of values of x for which y is decreasing.

.....
(1)

(Total for Question E1 is 5 marks)



- E2** Figure 8 shows a container in the shape of an open cylinder, with a base but no lid. The container has radius r cm and height h cm, and its capacity is $64\pi \text{ cm}^3$. The total area of metal used for the base and the curved side is $S \text{ cm}^2$. The manufacturer wants to use as little metal as possible.

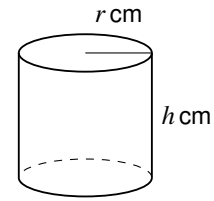


Diagram NOT accurately drawn

Figure 8

(a) Show that $S = \pi r^2 + \frac{128\pi}{r}$

(3)

- (b) Use calculus to find the value of r for which S is stationary.

.....
(1)

- (c) Find the minimum value of S , in terms of π , justifying that it is a minimum.

.....
(2)

(Total for Question E2 is 6 marks)



STRETCH

- E3** In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.

Figure 9 shows the curve C with equation

$$y = x^4 - 8x^2 + 3$$

The curve has a maximum point A on the y -axis and minimum points B and D , as shown in Figure 9.

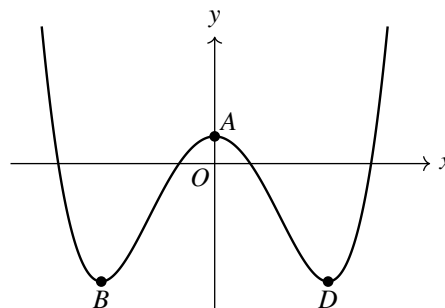


Figure 9

- (a) Using calculus, find the coordinates of A , B and D

.....
(3)

- (b) Use the second derivative to confirm the nature of A and of B

(2)

- (c) Find the set of values of k for which the equation $x^4 - 8x^2 + 3 = k$ has exactly four real solutions.

.....
(2)

- (d) State the set of values of x for which y is decreasing.

.....
(1)

(Total for Question E3 is 8 marks)

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