Binomial Expansion

Worked Solutions

Suitable for:

Pearson Edexcel International A-Level (iAL) — Pure Mathematics 4

Question 1 — Worked Solution

$$(1+x)^{-2} = 1 + (-2)x + \frac{(-2)(-3)}{2!}x^2 + \frac{(-2)(-3)(-4)}{3!}x^3 + \cdots$$

$$\therefore = 1 - 2x + 3x^2 - 4x^3 + \cdots$$

Valid for |x| < 1

Question 2 — Worked Solution

Replace x with 2x in the standard expansion of (1+x)$^{1/3}$

$$(1+2x)^{\frac{1}{3}} = 1 + \frac{1}{3}(2x) + \frac{\frac{1}{3}(-\frac{2}{3})}{2!}(2x)^2 + \frac{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})}{3!}(2x)^3 + \cdots$$

$$= 1 + \frac{2x}{3} + \frac{(-\frac{2}{9})}{2} \cdot 4x^2 + \frac{\frac{10}{27}}{6} \cdot 8x^3 + \cdots$$

$$\therefore = 1 + \frac{2x}{3} - \frac{4x^2}{9} + \frac{40x^3}{81} + \cdots$$

Valid for |2x| < 1 $\Rightarrow$ |x| < 1/2

Question 3 — Worked Solution

Replace x with –3x in $(1+x)^{-1/2}$

$$(1-3x)^{-\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)(-3x) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}(-3x)^2 + \cdots$$

$$= 1 + \frac{3x}{2} + \frac{\frac{3}{4}}{2} \cdot 9x^2 + \cdots$$

$$\therefore = 1 + \frac{3x}{2} + \frac{27x^2}{8} + \cdots$$

Valid for |3x| < 1 ⇒ |x| < 1/3

Question 4 — Worked Solution

Factor out 9 first

$$(9+x)^{\frac{1}{2}} = 9^{\frac{1}{2}}\left(1+\frac{x}{9}\right)^{\frac{1}{2}} = 3\left(1+\frac{x}{9}\right)^{\frac{1}{2}}$$

Expand using $(1+u)^{1/2}$ with $u = x/9$:

$$= 3\left[1+\frac{1}{2}\cdot\frac{x}{9}-\frac{1}{8}\cdot\frac{x^2}{81}+\cdots\right]$$

$$\therefore = 3+\frac{x}{6}-\frac{x^2}{216}+\cdots$$

Valid for $|x/9| < 1 \Rightarrow |x| < 9$

Question 5 — Worked Solution

Replace x with 3x in $(1+x)^{-2}$

$$(1+3x)^{-2} = 1 + (-2)(3x) + 3(3x)^2 + \cdots$$

$$\therefore = 1 - 6x + 27x^2 + \cdots$$

Valid for $|3x| < 1 \Rightarrow |x| < 1/3$

Question 6 — Worked Solution

Factor out 2 first

$$(2+3x)^{-\frac{1}{2}} = 2^{-\frac{1}{2}}\left(1+\frac{3x}{2}\right)^{-\frac{1}{2}} = \frac{\sqrt{2}}{2}\left(1+\frac{3x}{2}\right)^{-\frac{1}{2}}$$

Expand with u = 3x/2:

$$= \frac{\sqrt{2}}{2}\left[1 - \frac{3x}{4} + \frac{3}{8}\cdot\frac{9x^2}{4} - \frac{5}{16}\cdot\frac{27x^3}{8} + \cdots\right]$$

$$= \frac{\sqrt{2}}{2}\left[1 - \frac{3x}{4} + \frac{27x^2}{32} - \frac{135x^3}{128} + \cdots\right]$$

$$\therefore = \frac{\sqrt{2}}{2} - \frac{3\sqrt{2}}{8}x + \frac{27\sqrt{2}}{64}x^2 - \frac{135\sqrt{2}}{256}x^3 + \cdots$$

Valid for |3x/2| < 1 ⇒ |x| < 2/3

Question 7 — Worked Solution

Factor out 4 first: $4^{-3/2} = 1/8$

$$(4-x)^{-\frac{3}{2}} = 4^{-\frac{3}{2}}\left(1-\frac{x}{4}\right)^{-\frac{3}{2}} = \frac{1}{8}\left(1-\frac{x}{4}\right)^{-\frac{3}{2}}$$

Use $(1+u)^{-3/2} = 1 - (3/2)u + (15/8)u^2 - (35/16)u^3 + \ldots$ with $u = -x/4$:

$$= \frac{1}{8}\left[1 + \frac{3x}{8} + \frac{15x^2}{128} + \frac{35x^3}{1024} + \cdots\right]$$

$$\therefore = \frac{1}{8} + \frac{3x}{64} + \frac{15x^2}{1024} + \frac{35x^3}{8192} + \cdots$$

Valid for $|x/4| < 1 \Rightarrow |x| < 4$

Question 8 — Worked Solution

Factor out 8 first: $8^{1/3} = 2$

$$(8+6x)^{\frac{1}{3}} = 8^{\frac{1}{3}}\left(1+\frac{3x}{4}\right)^{\frac{1}{3}} = 2\left(1+\frac{3x}{4}\right)^{\frac{1}{3}}$$

Use $(1+u)^{1/3} = 1 + u/3 - u^2/9 + 5u^3/81 + \ldots$ with $u = 3x/4$:

$$= 2\left[1+\frac{x}{4}-\frac{x^2}{16}+\frac{5x^3}{192}+\cdots\right]$$

$$\therefore = 2+\frac{x}{2}-\frac{x^2}{8}+\frac{5x^3}{96}+\cdots$$

Valid for $|3x/4| < 1 \Rightarrow |x| < 4/3$

Question 9 — Worked Solution

(a) Five terms of $(9+x)^{1/2}$

From Q4 with two additional terms:

$$\therefore (9+x)^{\frac{1}{2}} = 3 + \frac{x}{6} - \frac{x^2}{216} + \frac{x^3}{3888} - \frac{5x^4}{279936} + \cdots$$

(b) Approximate √10

$\sqrt{10} = (9+1)^{1/2}$, so use $x = 1$:

$$\sqrt{10} \approx 3 + \frac{1}{6} - \frac{1}{216} + \frac{1}{3888} - \frac{5}{279936}$$

$$\approx 3 + 0.16667 - 0.00463 + 0.000257 - 0.0000179$$

$$\therefore \sqrt{10} \approx 3.16228 \text{ (5 d.p.)}$$

(c) Validity

$|x| = 1 < 9$, so $x = 1$ lies within the range of validity. ✓

Question 10 — Worked Solution

(a) Partial fractions

$$\frac{2x+7}{(1-x)(2+x)} \equiv \frac{A}{1-x} + \frac{B}{2+x}$$

$$2x+7 \equiv A(2+x) + B(1-x)$$

x = 1: 9 = 3A ⇒ A = 3

x = –2: 3 = 3B ⇒ B = 1

$$\therefore = \frac{3}{1-x} + \frac{1}{2+x}$$

(b) Series expansion

$$\frac{3}{1-x} = 3(1 + x + x^2 + \cdots) = 3 + 3x + 3x^2 + \cdots$$

$$\frac{1}{2+x} = \frac{1}{2} \cdot \frac{1}{1+x/2} = \frac{1}{2}\left(1 - \frac{x}{2} + \frac{x^2}{4} - \cdots\right) = \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \cdots$$

Add:

$$\therefore = \frac{7}{2} + \frac{11x}{4} + \frac{25x^2}{8} + \cdots$$

Valid for |x| < 1 (most restrictive)

Question 11 — Worked Solution

(a) Partial fractions

$$\frac{x^2+x+6}{(1-2x)(1+x)^2} \equiv \frac{A}{1-2x} + \frac{B}{1+x} + \frac{C}{(1+x)^2}$$

$$x^2+x+6 \equiv A(1+x)^2 + B(1-2x)(1+x) + C(1-2x)$$

x = 1/2: 1/4+1/2+6 = A(9/4) ⇒ 27/4 = 9A/4 ⇒ A = 3

x = –1: 1–1+6 = C(3) ⇒ C = 2

Compare x^2: 1 = A – 2B ⇒ B = 1

$$\therefore = \frac{3}{1-2x} + \frac{1}{1+x} + \frac{2}{(1+x)^2}$$

(b) Series expansion

$$\frac{3}{1-2x} = 3(1+2x+4x^2+8x^3+\cdots) = 3+6x+12x^2+24x^3+\cdots$$

$$\frac{1}{1+x} = 1-x+x^2-x^3+\cdots$$

$$\frac{2}{(1+x)^2} = 2(1-2x+3x^2-4x^3+\cdots) = 2-4x+6x^2-8x^3+\cdots$$

Add:

$$= (3+1+2)+(6-1-4)x+(12+1+6)x^2+(24-1-8)x^3+\cdots$$

$$\therefore = 6+x+19x^2+15x^3+\cdots$$

Valid for |x| < 1/2 (most restrictive)