eClassroom

A-LEVEL MATHEMATICS • FURTHER PURE 1

Proof

Worksheet

Pearson Edexcel International A Level (WFM01) • Calculator allowed: show every step

Free Proof worksheet for A-Level and International A-Level Maths, with full worked solutions

A Examples & Your turn page 2 → **B** Fluency core skills page 4 → **C** Reasoning explain it page 6 → **D** Problem solving page 7 → **E** Exam practice page 9

Tap a section to jump straight to it

By the end of this worksheet you can: prove sums by induction • prove divisibility results • prove powers of matrices • set out the basis, assumption, inductive step and conclusion • *Stretch:* recurrences, and using a proved result.

Which questions are for me?

CORE The methods every A-level student must secure. Start here.

STRETCH Divisibility with a second term, matrix powers with $(-1)^n$, and second-order recurrences.

Every section has both, clearly labelled. Questions get harder as you go, and every skill in Sections A to D is needed in Section E.

Track your progress

Section	B	C	D	E
Core marks	23	10	13	13
Stretch marks	16	14	17	9
My score				

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Name ..... Class ..... Date .....

A Worked examples & Your turn

Read each example carefully. Cover it up, then answer the question beside it using the same method.

CORE *Secure the method*

Worked example A1: a sum

Prove that $\sum_{r=1}^{n}(2r+1) = n(n+2)$ for $n \in \mathbb{Z}^+$

$n = 1$: LHS = 3, RHS = $1 \times 3 = 3$

Assume $\sum_{r=1}^{k}(2r+1) = k(k+2)$. Then $\sum_{r=1}^{k+1} = k(k+2) + (2k+3) = k^2 + 4k + 3 = (k+1)(k+3)$

true for $n = k+1$

Your turn A1

Prove that $\sum_{r=1}^{n}(3r-1) = \frac{1}{2}n(3n+1)$ for $n \in \mathbb{Z}^+$

Worked example A2: divisibility

Prove that $4^n - 1$ is divisible by 3 for $n \in \mathbb{Z}^+$

f(1) = 3

$f(k+1) = 4 \times 4^k - 1 = 4f(k) + 3$

a sum of multiples of 3

Your turn A2

Prove that $9^n - 1$ is divisible by 8 for $n \in \mathbb{Z}^+$

Worked example A3: a matrix power

Prove that $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 2^n & 2^n - 1 \\ 0 & 1 \end{pmatrix}$ for $n \in \mathbb{Z}^+$

$n = 1$: $\begin{pmatrix} 2 & 2-1 \\ 0 & 1 \end{pmatrix}$, the matrix itself

$$\begin{pmatrix} 2^k & 2^k - 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2^{k+1} & 2^k + 2^k - 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{k+1} & 2^{k+1} - 1 \\ 0 & 1 \end{pmatrix}$$

Your turn A3

Prove that $\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ 2^n - 1 & 2^n \end{pmatrix}$ for $n \in \mathbb{Z}^+$

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STRETCH

Worked example A4: a recurrence

$u_1 = 2$, $u_{n+1} = 2u_n - 1$. Prove that $u_n = 2^{n-1} + 1$

$u_1 = 2^0 + 1 = 2$; assume $u_k = 2^{k-1} + 1$: $u_{k+1} = 2(2^{k-1} + 1) - 1$

$u_{k+1} = 2^k + 1$

Your turn A4

$u_1 = 5$, $u_{n+1} = 3u_n - 4$. Prove that $u_n = 3^n + 2$

Worked example A5: two powers

Prove that $3^{2n} + 7$ is divisible by 8 for $n \in \mathbb{Z}^+$

$f(1) = 16$; $f(k+1) - f(k) = 9 \times 9^k - 9^k = 8 \times 9^k$

$f(k+1) = f(k) + 8 \times 9^k$

Your turn A5

Prove that $2^{n+2} + 3^{2n+1}$ is divisible by 7 for $n \in \mathbb{Z}^+$

Worked example A6: a sum of fractions

Prove that $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$ for $n \in \mathbb{Z}^+$

$n = 1$: $\frac{1}{2} = \frac{1}{2}$; $\frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)}$

$= \frac{k+1}{k+2}$

Your turn A6

Prove that $\sum_{r=1}^{n} r \times r! = (n+1)! - 1$ for $n \in \mathbb{Z}^+$

Worked example A7: a product of consecutive integers

Prove that $n^3 + 5n$ is divisible by 6 for $n \in \mathbb{Z}^+$

$f(1) = 6$; $f(k+1) = f(k) + 3k(k+1) + 6$, and $k(k+1)$ is even

$3k(k+1)$ is a multiple of 6

Your turn A7

Prove that $n^3 - n$ is divisible by 6 for $n \in \mathbb{Z}^+$

B Fluency

Each question is a designed sequence. From one part to the next, ask yourself: what changed, and what did it do to the answer?

CORE *Secure the method*

B1 Show that each statement is true for $n = 1$

(a) $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

(b) $7^n - 1$ is divisible by 6

(c) $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 2^n & 2^n - 1 \\ 0 & 1 \end{pmatrix}$

(d) $u_n = 2^n + 1$, where $u_1 = 3$

[4 marks]

B2 Write $f(k+1)$ in terms of $f(k)$ when

(a) $f(n) = 5^n - 1$

(b) $f(n) = 3^{2n} - 1$

(c) $f(n) = 2^{3n} - 1$

(d) $f(n) = 11^n - 6$

[4 marks]

B3 Show that

(a) $\frac{1}{2}k(k+1) + (k+1) = \frac{1}{2}(k+1)(k+2)$

(b) $\frac{1}{4}k^2(k+1)^2 + (k+1)^3 = \frac{1}{4}(k+1)^2(k+2)^2$

(c) $\frac{1}{6}k(k+1)(2k+1) + (k+1)^2 = \frac{1}{6}(k+1)(k+2)(2k+3)$

(d) $k^2 + (2k+1) = (k+1)^2$

[4 marks]

B4 $\mathbf{M} = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}$

(a) Find $\mathbf{M}^2$

(b) Find $\mathbf{M}^3$

(c) Suggest a formula for $\mathbf{M}^n$

(d) Prove your formula by induction.

[4 marks]

B5 $u_1 = 4$ and $u_{n+1} = 2u_n - 3$

(a) Find u_2, u_3 and u_4

(b) Suggest a formula for u_n

(c) Prove your formula by induction.

[3 marks]

B6 $f(n) = 8^n - 3^n$

(a) Find $f(1)$

(b) Show that $f(k+1) = 8f(k) + 5 \times 3^k$

(c) Hence prove that $f(n)$ is divisible by 5 for $n \in \mathbb{Z}^+$

(d) Check that $f(4)$ is divisible by 5

[4 marks]

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STRETCH

B7 $S_n = \sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)}$

(a) Show that $S_1 = \frac{1}{3}$

(b) Show that $\frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} = \frac{k+1}{2k+3}$

(c) Hence prove that $S_n = \frac{n}{2n+1}$

(d) Find S_{50}

[4 marks]

B8 $\mathbf{M} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$

(a) Show that $\mathbf{M}^n = \begin{pmatrix} 1 & 2^n - 1 \\ 0 & 2^n \end{pmatrix}$ for $n = 1$

(b) Write $\mathbf{M}^{k+1}$ as a product.

(c) Complete the proof by induction.

(d) Deduce $\det(\mathbf{M}^n)$

[4 marks]

B9 $f(n) = 7^n + 3n - 1$

(a) Show that f(1) is divisible by 9

(b) Show that $f(k+1) - 7f(k) = 9(1-2k)$

(c) Hence prove that $f(n)$ is divisible by 9 for $n \in \mathbb{Z}^+$

(d) Check f(3)

[4 marks]

B10 $u_1 = 1$, $u_2 = 5$ and $u_{n+2} = 5u_{n+1} - 6u_n$

(a) Find u_3

(b) Show that $u_n = 3^n - 2^n$ for $n = 1$ and $n = 2$

(c) Prove that $u_n = 3^n - 2^n$ for $n \in \mathbb{Z}^+$

(d) Explain why two cases are needed at the start.

[4 marks]

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C Reasoning

Show your working and explain your thinking. A correct answer without a reason does not score full marks.

CORE *Secure the method*

C1 Proving $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$, Amir starts "When $n = 1$ the result is true." Explain why this would lose a mark, and write it correctly. **[2 marks]**

C2 Bea's proof that $5^n - 1$ is divisible by 4 ends "so f$(k+1)$ is divisible by 4". Explain what is missing, and write a full conclusion. **[2 marks]**

C3 Cal starts his inductive step: "Assume the result is true for $n = k+1$"

Explain his error, and state what should be assumed and what must be shown. **[3 marks]**

C4 For f$(n) = 6^n - 1$, Dev shows f$(k+1) -$ f$(k) = 5 \times 6^k$ and stops.

Explain why this does not yet prove divisibility by 5, and complete the step. **[3 marks]**

STRETCH

C5 Eve checks that $n^2 + n + 41$ is prime for $n = 1, 2, 3, \ldots, 10$ and says this proves it is prime for every positive integer n

Explain her error, and give a value of n for which it fails. **[3 marks]**

C6 Fay writes $\mathbf{M}^{k+1} = \mathbf{M}^k + \mathbf{M}$ in a matrix induction.

Explain her error, and state the correct expression. **[3 marks]**

C7 To prove $n^3 - n$ is divisible by 6, Gus shows f$(k+1) =$ f$(k) + 3k(k+1)$ and says "$3k(k+1)$ is a multiple of 3, so f$(k+1)$ is divisible by 6". Explain what is missing, and complete the argument. **[3 marks]**

C8 Prove by induction that, for all positive integers n,

$$\sum_{r=1}^{n} (6r^2 - 2r) = 2n^2(n+1)$$

[5 marks]

D Problem solving

You decide the method. Read the whole question first, and show every step.

CORE *Secure the method*

D1 The function f is defined by $f(n) = 2^{2n+1} + 1$, where n is a positive integer.

Prove by induction that, for $n \in \mathbb{Z}^+$, $f(n)$ is divisible by 3, making $f(k+1)$ the subject in your inductive step.

[3 marks]

D2 The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}$. Prove by induction that, for all positive integers n,

$$\mathbf{M}^n = \begin{pmatrix} 4^n & 4^n - 1 \\ 0 & 1 \end{pmatrix}$$

You must show the basis case clearly and give a full conclusion. **[4 marks]**

D3 Prove by induction that, for all positive integers n,

$$\sum_{r=1}^{n} 3^r = \frac{3}{2}(3^n - 1)$$

You should show the values of both sides when $n = 1$ **[3 marks]**

D4 A sequence of numbers is defined by $u_1 = 7$ and $u_{n+1} = 4u_n - 9$, $n \geqslant 1$

Prove by induction that, for $n \in \mathbb{Z}^+$, $u_n = 4^n + 3$ **[3 marks]**

STRETCH

D5 The function f is defined by $f(n) = 7^n - 6n - 1$, where n is an integer.

Prove by induction that, for all integers $n \geqslant 2$, $f(n)$ is divisible by 36, explaining why the basis case is $n = 2$

[4 marks]

D6 The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$. Prove by induction that, for all positive integers n,

$$\mathbf{M}^n = \begin{pmatrix} (-1)^n & 1 - (-1)^n \\ 0 & 1 \end{pmatrix}$$

Hence write down $\mathbf{M}^{20}$ **[4 marks]**

D7 Prove by induction that, for all positive integers n,

$$\sum_{r=1}^{n} (2r-1)^3 = n^2(2n^2 - 1)$$

Hence find the value of $1^3 + 3^3 + 5^3 + \cdots + 19^3$ **[4 marks]**

D8 $\mathbf{M} = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$

(a) Prove by induction that, for $n \in \mathbb{Z}^+$, $\mathbf{M}^n = \begin{pmatrix} 1 & 1-2^n \\ 0 & 2^n \end{pmatrix}$

(3)

A square of area 3 square units is transformed by the matrix $\mathbf{M}^n$ onto a quadrilateral of area 192 square units.

(b) Using your answer to part (a), determine the value of n

(2)

[5 marks]

E Exam practice

Exam-style questions in the style of WFM01. Write your answers in the spaces, and show every step.

CORE *Exam questions for everyone*

E1 *In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.*

Prove by induction that, for all positive integers n,

$$\sum_{r=1}^{n}(3r^2-3r+1)=n^3$$

(6)

(Total for Question E1 is 6 marks)

E2 **(i)** The function f is defined for positive integers. Prove by induction that, for $n \in \mathbb{Z}^+$, $f(n)=3^{2n+2}+8n-9$ is divisible by 16

(4)

(ii) A sequence of numbers is defined by $u_1=2$ and $u_{n+1}=3u_n+2$, $n \geqslant 1$

Prove by induction that, for $n \in \mathbb{Z}^+$, $u_n=3^n-1$

(3)

(Total for Question E2 is 7 marks)

STRETCH

E3 The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$

(a) Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\mathbf{M}^n = \begin{pmatrix} 2n+1 & -4n \\ n & 1-2n \end{pmatrix}$$

(5)

(b) The point $P(1,0)$ is mapped by $\mathbf{M}^n$ onto the point $(41,20)$. Determine the value of n

(2)

(c) Show that substituting $n = -1$ into the formula in part (a) gives $\mathbf{M}^{-1}$

(2)

(Total for Question E3 is 9 marks)

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