



Numerical Methods

Worked Solutions

Suitable for:

Pearson Edexcel International A-Level (iAL) — Pure Mathematics 3



Question 1 — Worked Solution

Show root of $x^3 - 2x - 5$ in $[2, 3]$

$$f(2) = 8 - 4 - 5 = -1 < 0$$

$$f(3) = 27 - 6 - 5 = 16 > 0$$

Since f is continuous and $f(2) < 0 < f(3)$, by the sign change rule there is at least one root in the interval $(2, 3)$.



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Question 2 — Worked Solution

(a) Root in $[0, 1]$

$$f(0) = e^0 - 0 = 1 > 0$$

$$f(1) = e - 3 = -0.282... < 0$$

Sign change $\Rightarrow$ root in $(0, 1)$. ✓

(b) Root in $[1, 2]$

$$f(1) = e - 3 < 0 \quad (\text{shown above})$$

$$f(2) = e^2 - 6 = 1.389... > 0$$

Sign change $\Rightarrow$ root in $(1, 2)$. ✓





Question 3 — Worked Solution

Show $\cos x - x = 0$ has root in $[0.7, 0.8]$

Let $f(x) = \cos x - x$:

$$f(0.7) = \cos(0.7) - 0.7 = 0.7648... - 0.7 = 0.0648... > 0$$

$$f(0.8) = \cos(0.8) - 0.8 = 0.6967... - 0.8 = -0.1033... < 0$$

Sign change $\Rightarrow$ root in $(0.7, 0.8)$. ✓





Question 4 — Worked Solution

(a) Root in [1, 2]

$$f(1) = 1 - 3 + 1 = -1 < 0 \quad f(2) = 8 - 6 + 1 = 3 > 0$$

Sign change $\Rightarrow$ root in (1, 2). ✓

(b) Rearrangement

$$x^3 - 3x + 1 = 0 \Rightarrow x^3 = 3x - 1 \Rightarrow x = (3x - 1)^{\frac{1}{3}} \quad \checkmark$$

(c) Iteration from $x_1 = 1.5$

n	x_n
1	1.500000
2	1.518294
3	1.526189
4	1.529571
5	1.531016
6	1.531632
7	1.531894

$\therefore$ Root ≈ 1.532 (3 d.p.)



Question 5 — Worked Solution

(a) Root in [4, 5]

$$f(4) = 16 - 20 + 2 = -2 < 0 \quad f(5) = 25 - 25 + 2 = 2 > 0$$

Sign change $\Rightarrow$ root in (4, 5). ✓

(b) Rearrangement

$$x^2 - 5x + 2 = 0 \Rightarrow x^2 = 5x - 2 \Rightarrow x = \frac{5x - 2}{x} = 5 - \frac{2}{x} \quad \checkmark$$

(c) Iteration from $x_1 = 4.5$

n	x_n
1	4.500000
2	4.555556
3	4.560976
4	4.561497
5	4.561547
6	4.561552

$\therefore$ Root ≈ 4.562 (3 d.p.)



Question 6 — Worked Solution

(a) Root in [1, 2]

$$f(1) = 1 - 1 - 1 = -1 < 0 \quad f(2) = 8 - 2 - 1 = 5 > 0$$

Sign change $\Rightarrow$ root in (1, 2). ✓

(b) Both rearrangements

$$(i) \quad x^3 = x + 1 \Rightarrow x = (x + 1)^{\frac{1}{3}} \quad \checkmark$$

$$(ii) \quad x^3 = x + 1 \Rightarrow x = x^3 - 1 \quad \checkmark$$

(c) Test both from $x_1 = 1.5$

Formula (i): $x = (x+1)^{1/3}$

n	x_n
1	1.500000
2	1.357209
3	1.330861
4	1.325884
5	1.324939
6	1.324760

Formula (i) **converges** to 1.325 (3 d.p.). $|g'(x)| = (1/3)(x+1)^{-2/3} \approx 0.19 < 1 \checkmark$

Formula (ii): $x = x^3 - 1 \rightarrow x_2 = 2.375, x_3 = 12.4, x_4 \rightarrow \infty$ **Diverges**. $|g'(x)| = 3x^2 \approx 6.75 > 1$

$\therefore$ Root ≈ 1.325 using formula (i)



Question 7 — Worked Solution

(a) Iteration table

n	x_n
1	2.000000
2	2.250000
3	2.197531
4	2.207076
5	2.205289
6	2.205622

(b) Root to 2 d.p.

$\therefore \text{Root} \approx 2.21$





Question 8 — Worked Solution

(a) Equation $f(x) = 0$

The fixed point iteration $x = g(x)$ means $x = \ln(x+3)$, so $f(x) = x - \ln(x+3)$.

$$f(x) = x - \ln(x + 3) = 0$$

(b) Convergence

$$g(x) = \ln(x + 3) \Rightarrow g'(x) = \frac{1}{x + 3}$$

Near the root ($x \approx 1.5$): $|g'(1.5)| = 1/4.5 = 0.22 < 1$.

Since $|g'(x)| < 1$ near the root, the iteration converges. ✓

(c) Iterations from $x_1 = 1.5$

n	x_n
1	1.500000
2	1.504077
3	1.504983
4	1.505184
5	1.505229
6	1.505239

$\therefore$ Root ≈ 1.505 (3 d.p.)



Question 9 — Worked Solution

(a) Root in [1, 2]

$$f(1) = 2\sin(1) - 1 = 0.683... > 0$$

$$f(2) = 2\sin(2) - 2 = -0.181... < 0$$

Sign change $\Rightarrow$ root in (1, 2). ✓

(b) Rearrangement

$$2\sin x - x = 0 \Rightarrow x = 2\sin x \quad \checkmark$$

(c) Iteration from $x_1 = 1.5$

The iteration oscillates but converges since $|g'(\text{root})| = |2\cos(1.896)| \approx 0.64 < 1$:

n	x_n
1	1.500000
2	1.994990
3	1.822742
4	1.936859
5	1.867488
6	1.912618
7	1.884291
8	1.902523

Continuing until stable to 3 d.p.:

$$\therefore \text{Root} \approx 1.896 \text{ (3 d.p.)}$$





Question 10 — Worked Solution

(a) Root in [2, 3]

$$f(2) = e^2 - 8 = -0.611... < 0 \quad f(3) = e^3 - 12 = 8.086... > 0$$

Sign change $\Rightarrow$ root in (2, 3). ✓

(b) Rearrangement

$$e^x = 4x \Rightarrow x = \ln(4x) \quad \checkmark$$

(c) Iteration from $x_1 = 2$

n	x_n
1	2.000000
2	2.079442
3	2.118394
4	2.136952
5	2.145675
6	2.149749
7	2.151645
8	2.152527

$\therefore$ Root ≈ 2.153 (3 d.p.)



Question 11 — Worked Solution

(a) Root in [2, 3]

$$f(2) = 2 + \ln 2 - 3 = -0.307... < 0$$

$$f(3) = 3 + \ln 3 - 3 = 1.099... > 0$$

Sign change $\Rightarrow$ root in (2, 3). ✓

(b) Rearrangement

$$x + \ln x = 3 \Rightarrow \ln x = 3 - x \Rightarrow x = 3 - \ln x \quad \checkmark$$

(c) Iteration from $x_1 = 2.2$

n	x_n
1	2.200000
2	2.211543
3	2.206310
4	2.208679
5	2.207606
6	2.208092
7	2.207871
8	2.207971

$\therefore$ Root ≈ 2.208 (3 d.p.)

