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INTERNATIONAL A LEVEL MATHEMATICS • PURE 4 (WMA14)

Pure 4 Practice Paper 2

Worked Solutions

AIMING FOR A

Written in the style of the Pearson Edexcel International Advanced Level paper (WMA14/01) • 75 marks

Every question solved in full on the paper itself, the way your teacher would write it out

How to read these solutions. Each page is the real question page, with a full answer written on the answer lines.

Red pen

M1 A1

boxed

the working a full-marks answer shows, with a short note on why each step is taken

the mark earned on that line: M = method, A = accuracy, B = independent mark

the final answer to each part

Sit the paper first. These solutions are most useful after you have tried every question yourself under timed conditions. Mark your paper with the mark scheme, then use this booklet to see exactly how each lost mark could have been earned.

Q	Topic	Marks	Page
1	Improper partial fractions and a series	9	2
2	A binomial approximation and a proof	8	3
3	Area under a parametric curve	8	4
4	A draining tank	8	5
5	A minimum point and a volume	9	7
6	Partial fractions and an exact area	8	9
7	Implicit gradients and a proof	8	10
8	The shortest distance to a line	9	11
9	Series approximation of an integral	8	12

Question paper & mark scheme **Exam questions by topic** **More practice papers**

1.

$$f(x) = \frac{4x^2 + x + 4}{(1+2x)(1-x)}$$

(a) Express $f(x)$ in the form

$$A + \frac{B}{1+2x} + \frac{C}{1-x}$$

where A, B and C are constants to be found.

(3)

(b) Find the series expansion of $f(x)$, in ascending powers of x, up to and including the term in x^2, and state the range of values of x for which it is valid.

(4)

(c) Given that the coefficient of x^2 in the expansion of $(1+kx)f(x)$ is zero, find the value of the constant k.

(2)

(a) Degree 2 over degree 2, so there is a constant: x^2 terms give $4 = -2A \Rightarrow A = -2$ B1

$4x^2 + x + 4 \equiv -2(1 + 2x)(1 - x) + B(1 - x) + C(1 + 2x)$

$x = 1$: $9 = 3C \Rightarrow C = 3$; $x = -½$: $1 - ½ + 4 = 1.5B \Rightarrow B = 3$ M1

$f(x) = -2 + 3/(1 + 2x) + 3/(1 - x)$ A1

(b) $3(1 + 2x)^{-1} = 3(1 - 2x + 4x^2) = 3 - 6x + 12x^2$ M1

$3(1 - x)^{-1} = 3 + 3x + 3x^2$, and do not forget the -2 M1

$-2 + 3 + 3 + (-6 + 3)x + (12 + 3)x^2$ $f(x) = 4 - 3x + 15x^2 + \ldots$ A1

$(1 + 2x)^{-1}$ needs $|2x| < 1$, stricter than $|x| < 1$: $|x| < ½$ B1

(c) $(1 + kx)(4 - 3x + 15x^2 + \ldots)$: x^2 terms are $15x^2 + kx \times (-3x) \Rightarrow 15 - 3k = 0$ M1

$k = 5$ A1

(Total for Question 1 is 9 marks)

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2. **(a)** Find the binomial expansion of

$$\sqrt{1-8x} \qquad |x| < \frac{1}{8}$$

in ascending powers of x, up to and including the term in x^3, simplifying each coefficient.

(3)

(b) By substituting $x = 0.01$ into your expansion, find a rational approximation to $\sqrt{23}$, giving your answer as a fraction in its simplest form.

(2)

(c) Prove by contradiction that $\sqrt{23}$ is irrational.

(3)

(a) $(1-8x)^{1/2} = 1 + \frac{1}{2}(-8x) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2}(-8x)^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-3/2)}{6}(-8x)^3$ M1

$= 1 - 4x - \frac{1}{8}(64x^2) + \frac{1}{16}(-512x^3)$ A1

$\boxed{1 - 4x - 8x^2 - 32x^3}$ A1

(b) $x = 0.01$: $\sqrt{0.92} = \sqrt{92} \div 10 = 2\sqrt{23} \div 10 = \sqrt{23} \div 5$ (the factor 5 matters)

$\sqrt{23} \approx 5(1 - 0.04 - 0.0008 - 0.000032)$ M1

$5 \times 0.959168 = 4.79584$ $\boxed{\sqrt{23} \approx 14987/3125}$ A1

(c) Assume $\sqrt{23} = \frac{p}{q}$, where p and q are integers with no common factor B1

$p^2 = 23q^2$. 23 is prime and divides p^2, so 23 divides p.

$p = 23m \Rightarrow 23^2m^2 = 23q^2 \Rightarrow q^2 = 23m^2$, so 23 also divides q M1

So 23 is a common factor of p and q: contradiction. $\boxed{\sqrt{23} \text{ is irrational}}$ A1

(Total for Question 2 is 8 marks)

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3. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

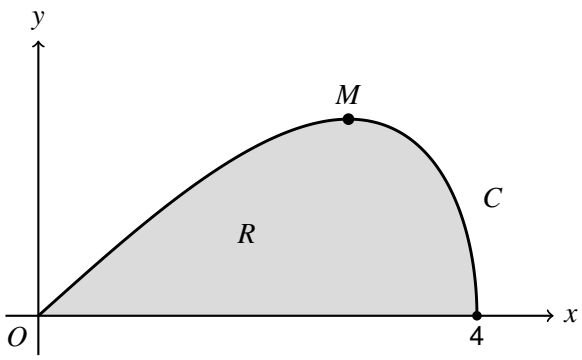


Figure 1

Figure 1 shows the curve C with parametric equations

$$x = 4\cos t, \qquad y = \sin 2t \qquad 0 \leqslant t \leqslant \frac{\pi}{2}$$

The point M is the maximum point on C.

(a) Find the exact coordinates of M.

(3)

The region R, shown shaded in Figure 1, is bounded by C and the x-axis.

(b) Find the exact area of R.

(5)

(a) $\frac{dy}{dt} = 2\cos 2t$, $\frac{dx}{dt} = -4\sin t$. At the maximum M, $\frac{dy}{dt} = 0$ M1

$\cos 2t = 0$ with $0 \leqslant 2t \leqslant \pi \Rightarrow 2t = \frac{\pi}{2} \Rightarrow t = \frac{\pi}{4}$ M1

$x = 4\cos\frac{\pi}{4} = 2\sqrt{2}$, $y = \sin\frac{\pi}{2} = 1$ $M(2\sqrt{2}, 1)$ A1

(b) Area $= \int y \frac{dx}{dt}\, dt = \int \sin 2t \times (-4\sin t)\, dt$ M1

$= \int -8\sin^2 t \cos t\, dt$ (using $\sin 2t = 2\sin t \cos t$) M1

$x = 0$ when $t = \frac{\pi}{2}$, and $x = 4$ when $t = 0$, so t runs from $\frac{\pi}{2}$ down to 0 A1

$[-\frac{8}{3}\sin^3 t]$ from $t = \frac{\pi}{2}$ to $t = 0$: $0 - (-\frac{8}{3})$ M1

Area of R = 8/3 (limits the other way give −8/3: not an area) A1

(Total for Question 3 is 8 marks)

4.

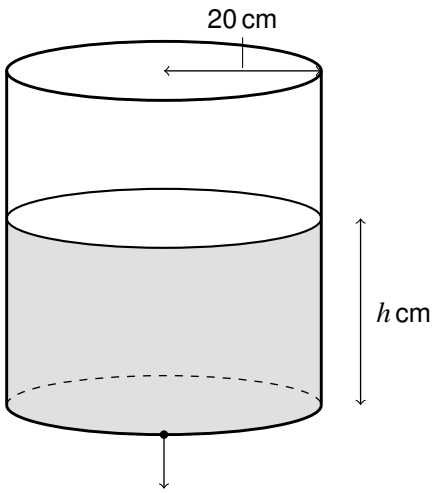


Figure 2

Figure 2 shows a tank in the shape of a vertical cylinder with radius 20 cm.

Water flows out of a small hole in the base of the tank. When the depth of the water is h cm, the water flows out at a rate of $80\pi\sqrt{h}\,\text{cm}^3$ per minute.

At time $t = 0$ minutes the depth of the water is 100 cm, and no water flows into the tank.

(a) Form and solve a differential equation to find h in terms of t.

(5)

(b) Sketch the graph of h against t, for the time until the tank is empty.

(1)

(c) Find, to the nearest tenth of a minute, the time taken for the depth of the water to halve.

(2)

(a) $V = \pi \times 20^2 \times h = 400\pi h \Rightarrow \frac{dV}{dt} = 400\pi\frac{dh}{dt}$ M1

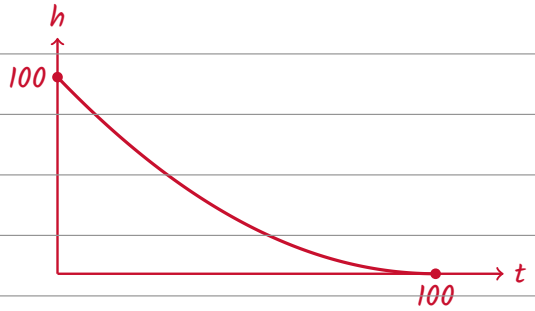
$400\pi\frac{dh}{dt} = -80\pi\sqrt{h} \Rightarrow \frac{dh}{dt} = -0.2\sqrt{h}$ (negative: the depth falls) A1

Separate: $\int h^{-1/2}\,dh = \int -0.2\,dt \Rightarrow 2\sqrt{h} = -0.2t + c$ M1

$t = 0, h = 100$: $2 \times 10 = c \Rightarrow c = 20$ M1

$\sqrt{h} = 10 - 0.1t$ $\boxed{h = (10 - 0.1t)^2}$ A1

(b) $h = 0$ when $t = 100$ (empty), and $\frac{dh}{dt} \to 0$ there, so the curve flattens.



B1

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Question 4 continued

(c) $h = 50$: $\sqrt{50} = 10 - 0.1t \Rightarrow t = 100 - 10\sqrt{50}$ M1

$= 100 - 70.71... = 29.29...$ $t = 29.3$ minutes (not 50: it drains fastest when full) A1

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(Total for Question 4 is 8 marks)

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5. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

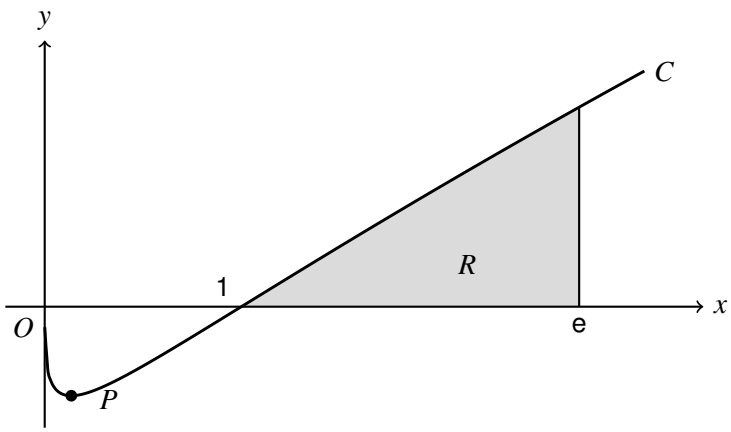

Figure 3

Figure 3 shows a sketch of the curve C with equation

$$y = \sqrt{x}\ln x \qquad x > 0$$

The point P is the minimum point on C.

(a) Find the exact coordinates of P.

(3)

The region R, shown shaded in Figure 3, is bounded by C, the x-axis and the line $x = e$

The region R is rotated through 2π radians about the x-axis to form a solid of revolution.

(b) Find the exact volume of the solid.

(6)

(a) $y = x^{1/2}\ln x$: $\frac{dy}{dx} = \frac{1}{2}x^{-1/2}\ln x + x^{1/2} \times \frac{1}{x} = (\ln x + 2) \div 2\sqrt{x}$ M1

$\frac{dy}{dx} = 0 \Rightarrow \ln x = -2 \Rightarrow x = e^{-2}$ M1

$y = e^{-1} \times (-2)$ $\boxed{P(e^{-2}, -2e^{-1})}$ A1

(b) C meets the x-axis at $x = 1$. $V = \pi\int y^2\,dx = \pi\int x(\ln x)^2\,dx$, from 1 to e B1

$u = (\ln x)^2$, $v = \frac{1}{2}x^2$: $\int x(\ln x)^2\,dx = \frac{1}{2}x^2(\ln x)^2 - \int \frac{1}{2}x^2 \times (2\ln x \div x)\,dx$

$= \frac{1}{2}x^2(\ln x)^2 - \int x\ln x\,dx$ M1

$u = \ln x$, $v = \frac{1}{2}x^2$: $\int x\ln x\,dx = \frac{1}{2}x^2\ln x - \int \frac{1}{2}x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2$ M1

So $\int x(\ln x)^2\,dx = \frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2\ln x + \frac{1}{4}x^2$ A1

From 1 to e: $(\frac{1}{2}e^2 - \frac{1}{2}e^2 + \frac{1}{4}e^2) - (0 - 0 + \frac{1}{4}) = \frac{1}{4}(e^2 - 1)$ M1

$\boxed{V = (\pi/4)(e^2 - 1)}$ A1

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Question 5 continued

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(Total for Question 5 is 9 marks)

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6. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

$$f(x) = \frac{2x^2 + 1}{x^2(2x + 1)} \qquad x > 0$$

(a) Express $f(x)$ in partial fractions.

(3)

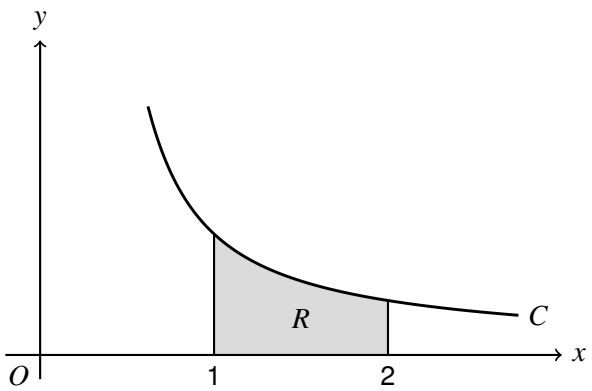

Figure 4

Figure 4 shows part of the curve with equation $y = f(x)$

The region R, shown shaded in Figure 4, is bounded by the curve, the x-axis and the lines $x = 1$ and $x = 2$

(b) Find the exact area of R, giving your answer in the form $p + \ln q$, where p and q are rational constants.

(5)

(a) $\frac{2x^2 + 1}{x^2(2x + 1)} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{2x + 1}$: $2x^2 + 1 \equiv Ax(2x + 1) + B(2x + 1) + Cx^2$

$x = 0$: $B = 1$ B1

$x = -\frac{1}{2}$: $1.5 = \frac{1}{4}C \Rightarrow C = 6$. x^2 terms: $2 = 2A + C \Rightarrow A = -2$ M1

$f(x) = -2/x + 1/x^2 + 6/(2x + 1)$ A1

(b) $\int \left(-\frac{2}{x} + x^{-2} + \frac{6}{2x + 1}\right) dx = -2 \ln x - \frac{1}{x} + 3 \ln(2x + 1)$ M1 A1

From 1 to 2: $(-2 \ln 2 - \frac{1}{2} + 3 \ln 5) - (0 - 1 + 3 \ln 3)$ M1

$= \frac{1}{2} + \ln 125 - \ln 4 - \ln 27 = \frac{1}{2} + \ln\frac{125}{108}$ M1

Area of $R = 1/2 + \ln(125/108)$ A1

(Total for Question 6 is 8 marks)

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7. The curve C has equation

$$y^3 + y = x^3 - x$$

(a) Find $\frac{dy}{dx}$ in terms of x and y.

(3)

(b) Find an equation of the tangent to C at the point on C where $x = 1$

(2)

(c) Prove by contradiction that there is no point on C at which the gradient of C is less than -1

(3)

(a) Differentiate with respect to x: $3y^2\frac{dy}{dx} + \frac{dy}{dx} = 3x^2 - 1$ M1

$(3y^2 + 1)\frac{dy}{dx} = 3x^2 - 1$ M1

$dy/dx = (3x^2 - 1)/(3y^2 + 1)$ A1

(b) $x = 1$: $y^3 + y = 0 \Rightarrow y(y^2 + 1) = 0 \Rightarrow y = 0$ only $(y^2 + 1 > 0)$. Gradient $= 2 \div 1 = 2$ M1

$y - 0 = 2(x - 1)$ $y = 2x - 2$ A1

(c) Assume there is a point on C where $(3x^2 - 1) \div (3y^2 + 1) < -1$ B1

$3y^2 + 1 > 0$, so multiplying by it keeps the sign: $3x^2 - 1 < -3y^2 - 1 \Rightarrow 3x^2 + 3y^2 < 0$ M1

But $3x^2 + 3y^2 \geqslant 0$ for all x, y: contradiction. No gradient is less than -1 A1

(Total for Question 7 is 8 marks)

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8. The line l has equation

$$\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

The point A has coordinates (6, 4, 1), and P is the point on l with parameter λ.

(a) Find AP^2 in terms of λ, simplifying your answer.

(2)

(b) Use calculus to find the shortest distance from A to l, and the coordinates of the point F on l that is closest to A.

(4)

(c) Verify that AF is perpendicular to l.

(1)

(d) Find the coordinates of the reflection of A in the line l.

(2)

(a) $P = (2 + \lambda, -1 + 2\lambda, 3 - 2\lambda)$, so $AP = (\lambda - 4, 2\lambda - 5, 2 - 2\lambda)$ M1

$AP^2 = (\lambda - 4)^2 + (2\lambda - 5)^2 + (2 - 2\lambda)^2$ $\boxed{AP^2 = 9\lambda^2 - 36\lambda + 45}$ A1

(b) $d(AP^2)/d\lambda = 18\lambda - 36 = 0$ (minimise AP^2, which is easier than AP) M1

$\lambda = 2$ (a minimum, as $9\lambda^2$ has a positive coefficient) A1

P at $\lambda = 2$: $\boxed{F(4, 3, -1)}$ A1

$AP^2 = 36 - 72 + 45 = 9$ $\boxed{\text{shortest distance} = 3}$ A1

(c) $AF = (-2, -1, -2)$: $(-2, -1, -2)\cdot(1, 2, -2) = -2 - 2 + 4 = 0$, so $AF \perp l$ ✓ B1

(d) F is the midpoint of A and its image: $2F - A = (8 - 6, 6 - 4, -2 - 1)$ M1

$\boxed{(2, 2, -3)}$ A1

(Total for Question 8 is 9 marks)

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9. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

(a) Find the binomial expansion of $(1+x)^{-\frac{1}{2}}$, in ascending powers of x, up to and including the term in x^2.

(2)

(b) Use your expansion to find an approximate value of

$$\int_0^{0.2} \frac{x}{\sqrt{1+x}}\,\mathrm{d}x$$

giving your answer to 5 significant figures.

(2)

(c) Use the substitution $u = 1+x$ to find the exact value of this integral, giving your answer in the form $a+b\sqrt{30}$, where a and b are rational.

(3)

(d) Find the percentage error in your answer to part (b).

(1)

(a) $(1+x)^{-1/2} = 1 + (-\frac{1}{2})x + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2}x^2 + \ldots$ M1

$1 - \frac{1}{2}x + (3/8)x^2$ A1

(b) $\int x(1 - \frac{1}{2}x + \frac{3}{8}x^2)\,dx = \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{3}{32}x^4$, from 0 to 0.2 M1

$0.02 - 0.0013333\ldots + 0.00015 = 0.0188166\ldots$ 0.018817 A1

(c) $u = 1 + x$: $du = dx$, $x = u - 1$, limits 1 to 1.2: $\int (u-1)u^{-1/2}\,du = \int (u^{1/2} - u^{-1/2})\,du$ M1

$[\frac{2}{3}u^{3/2} - 2u^{1/2}]$ from 1 to 1.2 $= (0.8\sqrt{1.2} - 2\sqrt{1.2}) - (\frac{2}{3} - 2)$ A1

$\sqrt{1.2} = \sqrt{30} \div 5$: $= -1.2\sqrt{30} \div 5 + \frac{4}{3}$ $4/3 - (6/25)\sqrt{30}$ A1

(d) Exact $= 0.0187992\ldots$: $(0.0188167 - 0.0187992) \div 0.0187992 \times 100$ 0.093% B1

(Total for Question 9 is 8 marks)

TOTAL FOR PAPER IS 75 MARKS

END

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