Integration

Worked Solutions

Suitable for:

Pearson Edexcel International A-Level (iAL) — Pure Mathematics 3

Question 1 — Worked Solution

(a) $\int \sin 3x \, dx = -\frac{\cos 3x}{3} + c$

(b) $\int e^{4x} \, dx = \frac{e^{4x}}{4} + c$

(c) $\int \frac{1}{2x+1} \, dx = \frac{\ln|2x+1|}{2} + c$

(d) $\int \cos \frac{x}{2} \, dx = 2\sin \frac{x}{2} + c$

Question 2 — Worked Solution

(a) $\int (3x+2)^5 \, dx = \frac{(3x+2)^6}{6 \times 3} + c = \frac{(3x+2)^6}{18} + c$

(b) $\int e^{2x-1} \, dx = \frac{e^{2x-1}}{2} + c$

(c) $\int \frac{1}{4x-3} \, dx = \frac{\ln|4x-3|}{4} + c$

Question 3 — Worked Solution

(a)

$$\int_0^{\pi/4} \cos 2x \, dx = \left[\frac{\sin 2x}{2}\right]_0^{\pi/4} = \frac{\sin(\pi/2)}{2} - \frac{\sin 0}{2} = \frac{1}{2}$$

(b)

$$\int_1^2 e^{3x} \, dx = \left[\frac{e^{3x}}{3}\right]_1^2 = \frac{e^6}{3} - \frac{e^3}{3}$$

$$\therefore = \frac{e^6 - e^3}{3}$$

Question 4 — Worked Solution

Show ∫sin²x dx = x/2 – sin2x/4 + c

From cos2x = 1–2sin²x: sin²x = (1–cos2x)/2

$$\int \sin^2 x \, dx = \int \frac{1-\cos 2x}{2} \, dx = \frac{x}{2} - \frac{\sin 2x}{4} + c \quad \checkmark$$

Evaluate ∫■^(π/2) sin²x dx

$$\left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{\pi/2} = \left(\frac{\pi}{4} - \frac{\sin \pi}{4}\right) - 0 = \frac{\pi}{4}$$

$$\therefore = \frac{\pi}{4}$$

Question 5 — Worked Solution

Evaluate ∫■^(π/3) cos^{2}x dx

Use cos2x = 2cos^{2}x – 1 ⇒ cos^{2}x = (1+cos2x)/2:

$$\int_0^{\pi/3} \cos^2 x \, dx = \int_0^{\pi/3} \frac{1+\cos 2x}{2} \, dx = \left[\frac{x}{2} + \frac{\sin 2x}{4}\right]_0^{\pi/3}$$

$$= \frac{\pi}{6} + \frac{\sin(2\pi/3)}{4} = \frac{\pi}{6} + \frac{\sqrt{3}/2}{4}$$

$$\therefore = \frac{\pi}{6} + \frac{\sqrt{3}}{8}$$

Question 6 — Worked Solution

Find ∫sinx cosx dx

From sin2x = 2sinx cosx: sinx cosx = sin2x/2

$$\int \sin x \cos x \, dx = \int \frac{\sin 2x}{2} \, dx = -\frac{\cos 2x}{4} + c$$

Question 7 — Worked Solution

Find $\int \tan^2 x \, dx$

From $\sec^2 x = 1 + \tan^2 x$: $\tan^2 x = \sec^2 x - 1$

$$\int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx = \tan x - x + c$$

Evaluate ∫■^(π/4) $\tan^2 x$ dx

$$[\tan x - x]_0^{\pi/4} = \left(1 - \frac{\pi}{4}\right) - 0$$

$$\therefore = 1 - \frac{\pi}{4}$$

Question 8 — Worked Solution

(a) $\int 2x(x^2+3)^4\,dx$

$\frac{d}{dx}[(x^2+3)^5] = 5(x^2+3)^4 \cdot 2x = 10x(x^2+3)^4$

$$\therefore \int 2x(x^2+3)^4\,dx = \frac{(x^2+3)^5}{5} + c$$

(b) $\int \sin x \cos^4 x\,dx$

$\frac{d}{dx}[-\cos^5 x] = -5\cos^4 x \cdot (-\sin x) = 5\sin x \cos^4 x$

$$\therefore \int \sin x \cos^4 x\,dx = -\frac{\cos^5 x}{5} + c$$

(c) $\int (2x+1)e^{x^2+x}\,dx$

$\frac{d}{dx}[e^{x^2+x}] = (2x+1)e^{x^2+x}$

$$\therefore \int (2x+1)e^{x^2+x}\,dx = e^{x^2+x} + c$$

Question 9 — Worked Solution

(a) ∫2x/(x^2+1) dx

Numerator is the derivative of the denominator.

$$\therefore \int \frac{2x}{x^2+1}\,\mathrm{d}x = \ln(x^2+1)+c$$

(b) ∫cosx/sinx dx

Numerator is the derivative of the denominator.

$$\therefore \int \frac{\cos x}{\sin x}\,\mathrm{d}x = \ln|\sin x|+c$$

(c) ∫(3x^2+2)/(x^3+2x) dx

d/dx[x^3+2x] = 3x^2+2   (exactly the numerator)

$$\therefore \int \frac{3x^2+2}{x^3+2x}\,\mathrm{d}x = \ln|x^3+2x|+c$$

Question 10 — Worked Solution

Area = ∫■^π $\sin^2 x$ dx

Use $\sin^2 x = (1 - \cos 2x)/2$:

$$\int_0^{\pi} \sin^2 x \, dx = \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{\pi}$$

$$= \left(\frac{\pi}{2} - \frac{\sin 2\pi}{4}\right) - \left(0 - \frac{\sin 0}{4}\right) = \frac{\pi}{2} - 0$$

$$\therefore \text{Area} = \frac{\pi}{2}$$

Question 11 — Worked Solution

$$\int_1^2\left(e^x + \frac{1}{x}\right)\mathrm{d}x = [e^x + \ln x]_1^2$$

$$= (e^2 + \ln 2) - (e^1 + \ln 1) = e^2 + \ln 2 - e$$

$$\therefore = e^2 - e + \ln 2$$

Question 12 — Worked Solution

(a) Show (cosx−sinx)^2 ≡ 1−sin2x

$$(\cos x - \sin x)^2 = \cos^2 x - 2\sin x\cos x + \sin^2 x$$

$$= (\cos^2 x + \sin^2 x) - 2\sin x\cos x = 1 - \sin 2x \quad ✓$$

(b) Evaluate ∫■^(π/6) (cosx−sinx)^2 dx

$$\int_0^{\pi/6} (1 - \sin 2x)\,dx = \left[x + \frac{\cos 2x}{2}\right]_0^{\pi/6}$$

$$= \left(\frac{\pi}{6} + \frac{\cos(\pi/3)}{2}\right) - \left(0 + \frac{1}{2}\right)$$

$$= \frac{\pi}{6} + \frac{1}{4} - \frac{1}{2}$$

$$\therefore = \frac{\pi}{6} - \frac{1}{4}$$

Question 13 — Worked Solution

Find $\int \sin(3x)\cos^4(3x)\,dx$

Recognise the form $f'(x)\cdot[f(x)]^n$:

$\frac{d}{dx}[\cos^5(3x)] = 5\cos^4(3x)\cdot(-\sin(3x))\cdot 3 = -15\sin(3x)\cos^4(3x)$

So $\sin(3x)\cos^4(3x) = -(1/15)\cdot\frac{d}{dx}[\cos^5(3x)]$

$$\therefore \int \sin(3x)\cos^4(3x)\,dx = -\frac{\cos^5(3x)}{15} + c$$

Question 14 — Worked Solution

Find $\int \sin 2x \cdot e^{\cos 2x}\,dx$

Recognise that $d/dx[\cos 2x] = -2\sin 2x$, so:

$d/dx[e^{\cos 2x}] = e^{\cos 2x} \cdot (-2\sin 2x) = -2\sin 2x \cdot e^{\cos 2x}$

Therefore: $\sin 2x \cdot e^{\cos 2x} = -(1/2) \cdot d/dx[e^{\cos 2x}]$

$$\therefore \int \sin 2x \cdot e^{\cos 2x}\,\mathrm{d}x = -\frac{e^{\cos 2x}}{2} + c$$

Question 15 — Worked Solution

Evaluate ∫■^(π/8) tan^2(2x) dx

Use $\sec^2(2x) = 1 + \tan^2(2x) \Rightarrow \tan^2(2x) = \sec^2(2x) - 1$:

$$\int_0^{\pi/8} \tan^2(2x)\,dx = \int_0^{\pi/8} (\sec^2(2x) - 1)\,dx$$

$$= \left[\frac{\tan(2x)}{2} - x\right]_0^{\pi/8}$$

$$= \left(\frac{\tan(\pi/4)}{2} - \frac{\pi}{8}\right) - 0 = \frac{1}{2} - \frac{\pi}{8}$$

$$\therefore = \frac{1}{2} - \frac{\pi}{8}$$