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Trigonometric Addition Formulae

Worked Solutions

Suitable for:

Pearson Edexcel International A-Level (iAL) — Pure Mathematics 3

Question 1 — Worked Solution

(a) Show sin 75° = (√6 + √2)/4

Write 75° = 45° + 30° and apply sin(A + B) = sin A cos B + cos A sin B:

$$\sin 75° = \sin 45° \cos 30° + \cos 45° \sin 30°$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$

$$\therefore = \frac{\sqrt{6} + \sqrt{2}}{4} \quad ✓$$

(b) Find cos 75°

Use cos(A + B) = cos A cos B – sin A sin B with A = 45°, B = 30°:

$$\cos 75° = \cos 45° \cos 30° - \sin 45° \sin 30°$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$$

$$\therefore = \frac{\sqrt{6} - \sqrt{2}}{4}$$

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Question 2 — Worked Solution

Show tan 75° = 2 + √3

Use tan(A + B) = (tan A + tan B)/(1 – tan A tan B) with A = 45°, B = 30°:

$$\tan 75° = \frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°} = \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}}$$

Multiply numerator and denominator by √3:

$$= \frac{\sqrt{3} + 1}{\sqrt{3} - 1}$$

Rationalise by multiplying by (√3 + 1)/(√3 + 1):

$$= \frac{(\sqrt{3} + 1)^2}{(\sqrt{3})^2 - 1^2} = \frac{3 + 2\sqrt{3} + 1}{2} = \frac{4 + 2\sqrt{3}}{2} = 2 + \sqrt{3} \quad ✓$$

Question 3 — Worked Solution

Show sin(x + π/3) + cos(x + π/6) ≡ √3 cos x

Expand both terms using addition formulae:

$$\sin\left(x+\frac{\pi}{3}\right)=\sin x\cos\frac{\pi}{3}+\cos x\sin\frac{\pi}{3}=\frac{1}{2}\sin x+\frac{\sqrt{3}}{2}\cos x$$

$$\cos\left(x+\frac{\pi}{6}\right)=\cos x\cos\frac{\pi}{6}-\sin x\sin\frac{\pi}{6}=\frac{\sqrt{3}}{2}\cos x-\frac{1}{2}\sin x$$

Add the two expressions:

$$=\frac{1}{2}\sin x+\frac{\sqrt{3}}{2}\cos x+\frac{\sqrt{3}}{2}\cos x-\frac{1}{2}\sin x$$

$$=\sqrt{3}\cos x=\text{RHS} \quad \checkmark$$

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Question 4 — Worked Solution

Given sinθ = 3/5, θ acute ⇒ cosθ = 4/5

(a) sin 2θ

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\frac{3}{5}\cdot\frac{4}{5}$$

$$\therefore = \frac{24}{25}$$

(b) cos 2θ

$$\cos 2\theta = 1 - 2\sin^2\theta = 1 - 2\cdot\frac{9}{25} = 1 - \frac{18}{25}$$

$$\therefore = \frac{7}{25}$$

(c) tan 2θ

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{\frac{24}{25}}{\frac{7}{25}}$$

$$\therefore = \frac{24}{7}$$

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Question 5 — Worked Solution

Prove $(\sin x + \cos x)^2 \equiv 1 + \sin 2x$

Expand the left-hand side:

$$\text{LHS} = \sin^2 x + 2\sin x\cos x + \cos^2 x$$

$$= (\sin^2 x + \cos^2 x) + 2\sin x\cos x$$

$$= 1 + \sin 2x = \text{RHS} \quad ✓$$

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Question 6 — Worked Solution

Prove cos 2x/(1 + sin 2x) ≡ (cos x – sin x)/(cos x + sin x)

Use $\cos 2x = \cos^2 x - \sin^2 x = (\cos x + \sin x)(\cos x - \sin x)$ and $\sin 2x = 2\sin x \cos x$:

$$\text{LHS} = \frac{(\cos x + \sin x)(\cos x - \sin x)}{1 + 2\sin x \cos x}$$

Note that $1 + 2\sin x \cos x = \sin^2 x + 2\sin x \cos x + \cos^2 x = (\sin x + \cos x)^2$:

$$= \frac{(\cos x + \sin x)(\cos x - \sin x)}{(\sin x + \cos x)^2} = \frac{\cos x - \sin x}{\cos x + \sin x} = \text{RHS} \quad ✓$$

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Question 7 — Worked Solution

Solve 3cos 2x – cos x = 2, 0° ≤ x ≤ 360°

Use $\cos 2x = 2\cos^2 x - 1$:

$$3(2\cos^2 x - 1) - \cos x = 2 \Rightarrow 6\cos^2 x - \cos x - 5 = 0$$

$$(6\cos x + 5)(\cos x - 1) = 0$$

$\cos x = 1 \rightarrow x = 0°, 360°$

$\cos x = -5/6 \rightarrow x = \cos^{-1}(-5/6) = 146.44°, \quad x = 360° - 146.44° = 213.56°$

$$\therefore x = 0°,\ 146.44°,\ 213.56°,\ 360°$$

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Question 8 — Worked Solution

Solve sin 2x = sin x, 0 ≤ x ≤ 2π

Use sin 2x = 2sin x cos x:

$$2\sin x\cos x = \sin x \;\Rightarrow\; \sin x(2\cos x - 1) = 0$$

sin x = 0 → x = 0, π, 2π

cos x = 1/2 → x = π/3, 5π/3

$$\therefore x = 0,\ \frac{\pi}{3},\ \pi,\ \frac{5\pi}{3},\ 2\pi$$

Question 9 — Worked Solution

Express 3sin x + 4cos x in the form R sin(x + α)

Expand R sin(x + α) = R sin x cos α + R cos x sin α. Comparing with 3sin x + 4cos x:

$$R\cos\alpha = 3 \quad \text{and} \quad R\sin\alpha = 4$$

$$R = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

$$\alpha = \arctan\left(\frac{4}{3}\right) = 53.13°$$

$$\therefore 3\sin x + 4\cos x = 5\sin(x + 53.13°)$$

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Question 10 — Worked Solution

Solve 3sin x + 4cos x = 3, 0° ≤ x ≤ 360°

From Q9: $5\sin(x + 53.13°) = 3 \Rightarrow \sin(x + 53.13°) = 3/5$

Let $u = x + 53.13°$. Range: $53.13° \le u \le 413.13°$

$u = \arcsin(3/5) = 36.87°$ (outside range — add 360°) $\Rightarrow u = 396.87°$

$$u = 180° - 36.87° = 143.13°$$

But 396.87° is in range. Convert back: $x = u - 53.13°$

$$x = 143.13° - 53.13° = 90°$$

$$x = 396.87° - 53.13° = 343.74°$$

$$\therefore x = 90° \text{ or } x = 343.74°$$

Question 11 — Worked Solution

(a) Express 5cos x – 12sin x as R cos(x + α)

Expand R cos(x + α) = R cos x cos α – R sin x sin α. Comparing with 5cos x – 12sin x:

$$R\cos\alpha = 5 \quad \text{and} \quad R\sin\alpha = 12$$

$$R = \sqrt{25+144} = \sqrt{169} = 13$$

$$\alpha = \arctan\left(\frac{12}{5}\right) = 67.38°$$

$$\therefore 5\cos x - 12\sin x = 13\cos(x + 67.38°)$$

(b) Maximum and minimum values

Maximum of cos(x + 67.38°) = 1, minimum = –1.

Maximum value = 13, Minimum value = –13

(c) Smallest positive x for maximum

Maximum when cos(x + 67.38°) = 1, i.e. x + 67.38° = 0° ⇒ x = –67.38° (not positive).

Next: x + 67.38° = 360° ⇒ x = 292.62°.

Smallest positive x = 292.62°

Question 12 — Worked Solution

Prove sin(x+y) + sin(x−y) ≡ 2sin x cos y

Expand each term using the addition formula:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$

Add:

$$\sin(x+y) + \sin(x-y) = 2\sin x \cos y = \text{RHS} \quad \checkmark$$

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Question 13 — Worked Solution

Prove $\cos 3x \equiv 4\cos^3 x - 3\cos x$

Write cos 3x = cos(2x + x) and apply the addition formula:

$$\cos(2x + x) = \cos 2x \cos x - \sin 2x \sin x$$

Substitute $\cos 2x = 2\cos^2 x - 1$ and $\sin 2x = 2\sin x \cos x$:

$$= (2\cos^2 x - 1)\cos x - (2\sin x \cos x)\sin x$$

$$= 2\cos^3 x - \cos x - 2\sin^2 x \cos x$$

Use $\sin^2 x = 1 - \cos^2 x$:

$$= 2\cos^3 x - \cos x - 2(1 - \cos^2 x)\cos x$$

$$= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x$$

$$= 4\cos^3 x - 3\cos x = \text{RHS} \quad \checkmark$$