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INTERNATIONAL A LEVEL MATHEMATICS • FURTHER PURE 1 (WFM01)

Further Pure 1 Practice Paper 2

Worked Solutions

AIMING FOR A

Written in the style of the Pearson Edexcel International Advanced Level paper (WFM01/01) • 75 marks

Every question solved in full on the paper itself, the way your teacher would write it out

How to read these solutions. Each page is the real question page, with a full answer written on the answer lines.

Red pen

M1 A1

boxed

the working a full-marks answer shows, with a short note on why each step is taken

the mark earned on that line: M = method, A = accuracy, B = independent mark

the final answer to each part

Sit the paper first. These solutions are most useful after you have tried every question yourself under timed conditions. Mark your paper with the mark scheme, then use this booklet to see exactly how each lost mark could have been earned.

Q	Topic	Marks	Page
1	A matrix with a parameter and an inverse	7	2
2	A cubic: a numerical root, then complex roots	8	3
3	Complex roots and a new quadratic	8	4
4	A series and an equation in n	9	5
5	A tangent to a parabola and the directrix	9	6
6	A combined transformation and area	9	8
7	A complex equation and modulus-argument form	8	9
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Question paper & mark scheme | Exam questions by topic | More practice papers

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1.

$$\mathbf{M} = \begin{pmatrix} k & 2 \\ 3 & k+1 \end{pmatrix}$$

where k is a constant.

(a) Find the values of k for which $\mathbf{M}$ is singular.

(3)

(b) Given that $\mathbf{M}$ is non-singular, find $\mathbf{M}^{-1}$ in terms of k

(2)

Given that $k = 4$, the transformation represented by $\mathbf{M}$ maps the point P onto the point (10, 18)

(c) Use your answer to part (b) to find the coordinates of P

(2)

(a) $\det M = k(k+1) - 2 \times 3 = k^2 + k - 6$ (singular means $\det M = 0$) M1

$k^2 + k - 6 = 0 \Rightarrow (k+3)(k-2) = 0$ M1

$\boxed{k = 2 \text{ or } k = -3}$ A1

(b) Swap the leading diagonal, change the signs of the other two, divide by det: M1

$M^{-1} = \frac{1}{k^2 + k - 6}\begin{pmatrix} k+1 & -2 \\ -3 & k \end{pmatrix}$ A1

(c) $k = 4$: $\det = 16 + 4 - 6 = 14$. $P = M^{-1} \times$ image (undo the transformation) M1

$\frac{1}{14}\begin{pmatrix} 5 & -2 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 10 \\ 18 \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 50 - 36 \\ -30 + 72 \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 14 \\ 42 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ $\boxed{P(1, 3)}$ A1

Check: $4 \times 1 + 2 \times 3 = 10$ ✓ $3 \times 1 + 5 \times 3 = 18$ ✓

(Total for Question 1 is 7 marks)

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2.

$$f(x) = x^3 - 2x^2 + 5x - 10$$

(a) Show that the equation $f(x) = 0$ has a root α in the interval [1.5, 2.5]

(2)

(b) Taking 1.5 as a first approximation to α, apply the Newton–Raphson process once to $f(x)$ to find a second approximation to α, giving your answer to 3 decimal places.

(3)

(c) Show that $\alpha = 2$ exactly, and hence find the other two roots of the equation $f(x) = 0$

(3)

(a) $f(1.5) = 3.375 - 4.5 + 7.5 - 10 = -3.625$ $f(2.5) = 5.625$ M1

Sign change, and f is continuous (a polynomial), so there is a root α in [1.5, 2.5] A1

(b) $f'(x) = 3x^2 - 4x + 5$, $f'(1.5) = 6.75 - 6 + 5 = 5.75$ B1

$x_1 = 1.5 - \frac{f(1.5)}{f'(1.5)} = 1.5 - \frac{-3.625}{5.75} = 1.5 + 0.6304... = 2.1304...$ M1

$\boxed{2.130}$ A1

(c) $f(2) = 8 - 8 + 10 - 10 = 0$, so $(x - 2)$ is a factor and $\alpha = 2$ exactly B1

$f(x) = (x - 2)(x^2 + 5)$ (compare coefficients: x^2 term $-2x^2$ ✓, constant -10 ✓) M1

$x^2 + 5 = 0 \Rightarrow x^2 = -5 \Rightarrow \boxed{x = \pm\sqrt{5}\,i}$ (not real, so 2 is the only real root) A1

(Total for Question 2 is 8 marks)

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3. The roots of the equation

$$3x^2 - 2x + 4 = 0$$

are α and β

(a) Write down the value of $\alpha + \beta$ and the value of $\alpha\beta$

(1)

(b) Without solving the equation, find the exact value of $\alpha^3 + \beta^3$

(3)

(c) Find a quadratic equation, with integer coefficients, whose roots are

$$\frac{\alpha}{\beta} \quad \text{and} \quad \frac{\beta}{\alpha}$$

(4)

(a) $\alpha + \beta = \frac{2}{3}$ $(-b \div a)$ $\quad \alpha\beta = \frac{4}{3}$ $(c \div a)$ B1

(b) $(\alpha + \beta)^3 = \alpha^3 + \beta^3 + 3\alpha\beta(\alpha + \beta)$, so $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$ M1

$= \frac{8}{27} - 3 \times \frac{4}{3} \times \frac{2}{3} = \frac{8}{27} - \frac{8}{3} = \frac{8}{27} - \frac{72}{27}$ M1

$\boxed{-\frac{64}{27}}$ A1

(c) Sum $= \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta}$, $\quad \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \frac{4}{9} - \frac{8}{3} = -\frac{20}{9}$ M1

Sum $= -\frac{20}{9} \div \frac{4}{3} = -\frac{20}{9} \times \frac{3}{4} = -\frac{5}{3}$ A1

Product $= \frac{\alpha}{\beta} \times \frac{\beta}{\alpha} = 1$ B1

$x^2 - (\text{sum})x + \text{product} = 0$: $\quad x^2 + \frac{5}{3}x + 1 = 0$, $\times 3$: $\quad \boxed{3x^2 + 5x + 3 = 0}$ A1

(Total for Question 3 is 8 marks)

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4. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

(a) Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that

$$\sum_{r=1}^{n} r(2r-1) = \tfrac{1}{6}n(n+1)(an+b)$$

where a and b are integers to be found.

(4)

(b) Hence find the value of n for which

$$\sum_{r=1}^{n} r(2r-1) = \sum_{r=1}^{n} \left(r^2+40\right)$$

(5)

(a) $\sum r(2r-1) = 2\sum r^2 - \sum r = \frac{2}{6}n(n+1)(2n+1) - \frac{1}{2}n(n+1)$ M1

$= \frac{1}{6}n(n+1)[2(2n+1) - 3]$ (take out the common factor $\frac{1}{6}n(n+1)$) M1

$= \frac{1}{6}n(n+1)(4n-1)$ A1

$a = 4,\ b = -1$ A1

(b) $\sum (r^2+40) = \sum r^2 + \sum 40 = \frac{1}{6}n(n+1)(2n+1) + 40n$ (40 added n times) M1 B1

$\frac{1}{6}n(n+1)(4n-1) - \frac{1}{6}n(n+1)(2n+1) = 40n \Rightarrow \frac{1}{6}n(n+1)(2n-2) = 40n$ M1

$n \neq 0$, so divide by n: $\frac{1}{3}(n+1)(n-1) = 40 \Rightarrow n^2 - 1 = 120 \Rightarrow n^2 = 121$ A1

$n = 11$ or $n = -11$; n is a positive integer, so reject -11 and 0: $n = 11$ A1

(Total for Question 4 is 9 marks)

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5.

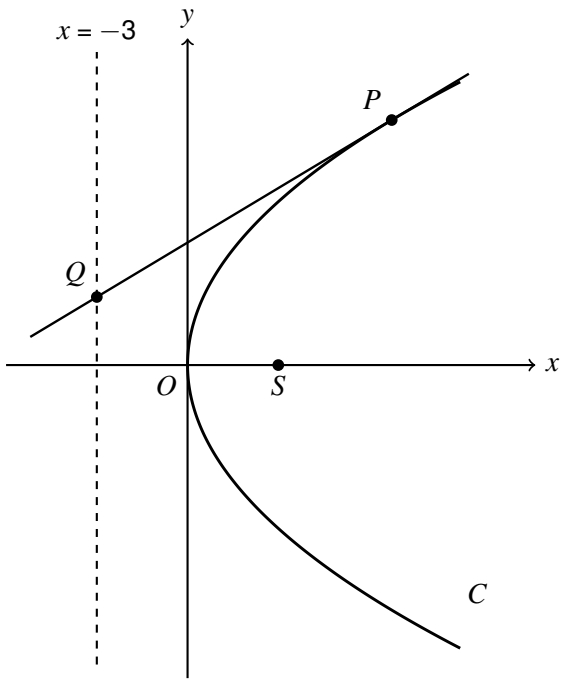

Figure 1

Figure 1 shows part of the parabola C with equation $y^2 = 12x$. The point S is the focus of C and the line with equation $x = -3$ is the directrix of C.

The point $P\left(3p^2, 6p\right)$, $p > 0$, lies on C. The tangent to C at P meets the directrix at the point Q

(a) Using calculus, find an equation of the tangent to C at P

(3)

(b) Find, in terms of p, the coordinates of Q

(2)

Given that the distance SP is 15

(c) find the value of p

(2)

(d) find the area of the triangle PSQ

(2)

(a) $y^2 = 12x$: $2y\frac{dy}{dx} = 12$, $\frac{dy}{dx} = \frac{6}{y} = \frac{6}{6p} = \frac{1}{p}$ at P M1 A1

$y - 6p = \frac{1}{p}(x - 3p^2) \Rightarrow py - 6p^2 = x - 3p^2 \Rightarrow \boxed{py = x + 3p^2}$ A1

(b) Q is on the directrix, so $x = -3$: $py = -3 + 3p^2 \Rightarrow y = \frac{3p^2 - 3}{p}$ M1

$\boxed{Q(-3, (3p^2 - 3)/p)}$ or $Q(-3, 3p - 3/p)$ A1

(c) $S(3, 0)$. Focus-directrix property: SP = distance from P to $x = -3 = 3p^2 + 3$ M1

$3p^2 + 3 = 15 \Rightarrow p^2 = 4 \Rightarrow p = 2$ $(p > 0)$ $\boxed{p = 2}$ A1

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Question 5 continued

(d) $P(12, 12)$, $S(3, 0)$, $Q\left(-3, \frac{12-3}{2}\right) = (-3, 4.5)$

Gradient $SP = \frac{12}{9} = \frac{4}{3}$, gradient $SQ = \frac{4.5}{-6} = -\frac{3}{4}$; product -1, so $\angle PSQ = 90°$ M1

$SQ = \sqrt{6^2 + 4.5^2} = 7.5$, Area $= \frac{1}{2} \times 15 \times 7.5 =$ 56.25 A1

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(Total for Question 5 is 9 marks)

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6. The transformation R is a reflection in the line $y = -x$

(a) Write down the matrix $\mathbf{R}$ that represents R

(1)

$$\mathbf{S} = \begin{pmatrix} k & -1 \\ 1 & 2 \end{pmatrix}$$

where k is a positive constant. The transformation represented by $\mathbf{S}$ is followed by the transformation R, and the combined transformation is represented by the matrix $\mathbf{C}$

(b) Find $\mathbf{C}$ in terms of k

(2)

(c) Find $\mathbf{C}^{-1}$ in terms of k

(2)

A triangle T has area 3 cm^2. The image of T under the transformation represented by $\mathbf{C}$ has area 33 cm^2

(d) Find the value of k

(2)

(e) Using this value of k, find the coordinates of the point that is mapped by $\mathbf{C}$ onto the point $(-8, -7)$

(2)

(a) $(1, 0) \to (0, -1)$ and $(0, 1) \to (-1, 0)$: $R = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ B1

(b) S first, then R, so $C = RS$ (the first transformation goes on the right) M1

$C = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} k & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ -k & 1 \end{pmatrix}$ A1

(c) $\det C = (-1)(1) - (-2)(-k) = -1 - 2k$ M1

$C^{-1} = \frac{1}{-1-2k}\begin{pmatrix} 1 & 2 \\ k & -1 \end{pmatrix}$ A1

(d) Area scale factor $= |\det C| = 2k + 1$ $(k > 0)$: $2k + 1 = \frac{33}{3} = 11$ M1

$k = 5$ (det C = −11: the reflection makes it negative, the area is still positive) A1

(e) $C^{-1} = \frac{1}{-11}\begin{pmatrix} 1 & 2 \\ 5 & -1 \end{pmatrix}$: $\frac{1}{-11}\begin{pmatrix} -8 - 14 \\ -40 + 7 \end{pmatrix} = \frac{1}{-11}\begin{pmatrix} -22 \\ -33 \end{pmatrix}$ M1

$(2, 3)$ Check: $-2 - 6 = -8$ ✓, $-10 + 3 = -7$ ✓ A1

(Total for Question 6 is 9 marks)

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7. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

The complex number z satisfies

$$3z - 2z^* = \frac{-12+8\mathrm{i}}{1+\mathrm{i}}$$

where z^* is the complex conjugate of z

(a) Find z, giving your answer in the form $a+b\mathrm{i}$, where a and b are real.

(4)

(b) Express z in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$ and $-\pi < \theta \leqslant \pi$, giving r and θ as exact values.

(2)

(c) Find z^2, and show z and z^2 on a single Argand diagram.

(2)

(a) $\times \frac{1-i}{1-i}$: $\frac{(-12+8i)(1-i)}{(1+i)(1-i)} = \frac{-12+12i+8i+8}{1+1} = \frac{-4+20i}{2} = -2+10i$ M1 A1

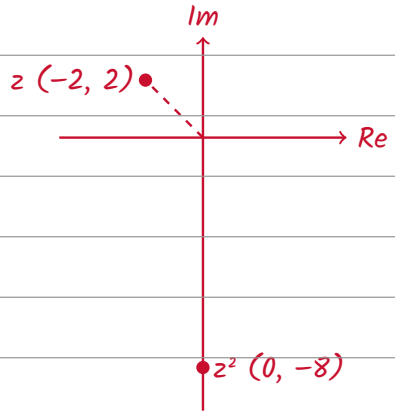
$z = a+bi$, $z^* = a-bi$: $3a+3bi-2a+2bi = a+5bi = -2+10i$ M1

$a = -2$, $5b = 10 \Rightarrow b = 2$: $\boxed{z = -2+2i}$ A1

(b) $r^2 = (-2)^2 + 2^2 = 8$, $r = 2\sqrt{2}$; z is in the 2nd quadrant, so $\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ M1

$\boxed{z = 2\sqrt{2}(\cos(3\pi/4) + i\sin(3\pi/4))}$ A1

(c) $z^2 = (-2+2i)^2 = 4 - 8i + 4i^2 = 4 - 8i - 4 = \boxed{-8i}$ B1



B1

(Total for Question 7 is 8 marks)

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8.

$$\mathbf{M} = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}$$

(a) Prove by induction that, for all positive integers n,

$$\mathbf{M}^n = \begin{pmatrix} 3^n & 3^n - 1 \\ 0 & 1 \end{pmatrix}$$

(6)

(b) Hence find $(\mathbf{M}^n)^{-1}$ in terms of n

(2)

(a) $n = 1$: $\begin{pmatrix} 3^1 & 3^1 - 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} = M$, so true for $n = 1$ **B1**

Assume true for $n = k$: $M^{k+1} = M^k M = \begin{pmatrix} 3^k & 3^k - 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}$ **M1**

$= \begin{pmatrix} 3 \times 3^k & 2 \times 3^k + 3^k - 1 \\ 0 & 1 \end{pmatrix}$ **A1**

$2 \times 3^k + 3^k = 3 \times 3^k = 3^{k+1}$ **M1**

$M^{k+1} = \begin{pmatrix} 3^{k+1} & 3^{k+1} - 1 \\ 0 & 1 \end{pmatrix}$, which is the result with $n = k + 1$ **A1**

True for $n = 1$, and if true for $n = k$ then true for $n = k + 1$, so true for all $n \geqslant 1$. **A1**

(b) $\det M^n = 3^n \times 1 - 0 \times (3^n - 1) = 3^n$ **M1**

$(M^n)^{-1} = \frac{1}{3^n}\begin{pmatrix} 1 & 1 - 3^n \\ 0 & 3^n \end{pmatrix}$ (or $\begin{pmatrix} 3^{-n} & 3^{-n} - 1 \\ 0 & 1 \end{pmatrix}$, the same matrix) **A1**

(Total for Question 8 is 8 marks)

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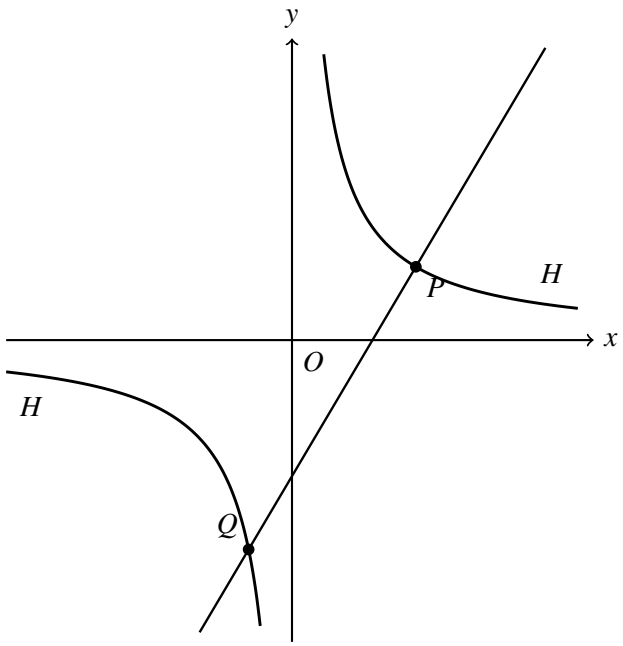

Figure 2

The rectangular hyperbola H has equation $xy = 9$. The point $P\left(3p, \frac{3}{p}\right)$, $p > 0$, lies on H. The normal to H at P meets H again at the point Q, as shown in Figure 2.

(a) Show that an equation of the normal to H at P is

$$p^3x - py = 3p^4 - 3$$

(4)

(b) Find, in terms of p, the coordinates of Q

(4)

(c) Write down the coordinates of Q when P is the point $\left(6, \frac{3}{2}\right)$

(1)

(a) $y = 9x^{-1}$, $\frac{dy}{dx} = -9x^{-2} = -\frac{9}{9p^2} = -\frac{1}{p^2}$ at P, so the normal gradient is p^2 M1 A1

$y - \frac{3}{p} = p^2(x - 3p)$ M1

$\times p$: $py - 3 = p^3x - 3p^4$, so $p^3x - py = 3p^4 - 3$ as required A1

(b) Q is on H, so let $Q = (3q, \frac{3}{q})$: $p^3(3q) - p(\frac{3}{q}) = 3p^4 - 3$ M1

$\times \frac{q}{3}$: $p^3q^2 - p = p^4q - q \Rightarrow p^3q^2 + (1 - p^4)q - p = 0$ A1

$q = p$ must be a root (the normal passes through P), so product of roots $= \frac{-p}{p^3} = -\frac{1}{p^2}$

$p \times q = -\frac{1}{p^2} \Rightarrow q = -\frac{1}{p^3}$ (or $(q - p)(p^3q + 1) = 0$) M1

$Q(-\frac{3}{p^3}, -3p^3)$ A1

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Question 9 continued

(c) $3p = 6$, so $p = 2$: $Q(-\frac{3}{8}, -24)$ $\boxed{(-0.375, -24)}$ B1

Check: $(-\frac{3}{8})(-24) = 9$ ✓ and $8(-\frac{3}{8}) - 2(-24) = 45 = 3(16) - 3$ ✓

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(Total for Question 9 is 9 marks)

TOTAL FOR PAPER IS 75 MARKS

END

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