



eClassroom

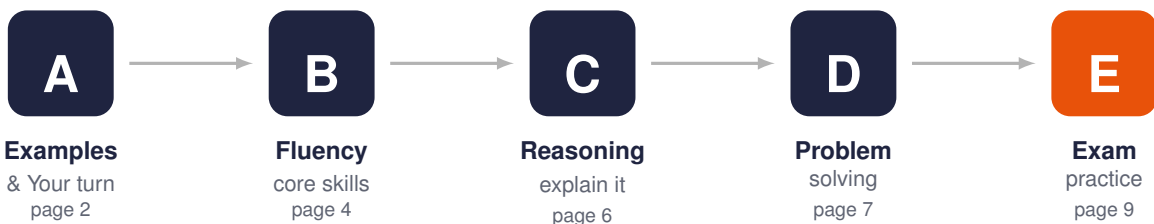
A-LEVEL MATHEMATICS • PURE 4

Proof

Worksheet

Pearson Edexcel International A Level (WMA14) & A Level (9MA0) • Calculator allowed: show every step

Free Proof worksheet for A-Level and International A-Level Maths, with full worked solutions



Tap a section to jump straight to it

By the end of this worksheet you can: write the assumption for any statement and finish with a contradiction and a conclusion • prove that surds and cube roots are irrational • prove that there are infinitely many primes • use algebra, not a calculator or a sketch, to prove that curves do not meet or have no stationary points • *Stretch:* choose cases by remainders, state the sign of a factor before multiplying, and build a proof from a lemma.

Which questions are for me?

CORE The methods every A-level student must secure. Start here.

STRETCH Cube roots, Euclid's argument for primes, inequalities where a sign decides the step, and curves whose sketch looks stationary.

Every section has both, clearly labelled. Questions get harder as you go, and every skill in Sections A to D is needed in Section E.

Track your progress

Section	B	C	D	E
Core marks	23	10	13	11
Stretch marks	16	14	17	8
My score				

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A

Worked examples & Your turn

Read each example carefully. Cover it up, then answer the question beside it using the same method.

CORE

Secure the method

Worked example A1: the method

n is an integer. Prove by contradiction that if n^2 is even, then n is even.

Assume the statement is false: n^2 is even and n is odd.

$$n = 2k + 1: n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

So n^2 is odd, which contradicts n^2 being even.

contradiction, so if n^2 is even, n is even

Your turn A1

n is an integer. Prove by contradiction that if $n^2 + 4n$ is odd, then n is odd.

Worked example A2: irrational numbers

Prove by contradiction that $\sqrt{2}$ is irrational.

Assume $\sqrt{2} = \frac{a}{b}$, where a and b are integers with no common factors.

$$a^2 = 2b^2, \text{ so } a^2 \text{ is even, so } a \text{ is even: } a = 2m$$

$$4m^2 = 2b^2 \Rightarrow b^2 = 2m^2, \text{ so } b^2 \text{ is even and } b \text{ is even}$$

2 is a common factor: a contradiction

Your turn A2

Prove by contradiction that $\sqrt{5}$ is irrational.

You may use the fact that if a^2 is a multiple of 5, then a is a multiple of 5

Worked example A3: curves that do not meet

Prove by contradiction that the line $y = 2x - 3$ does not meet the curve $y = x^2 + 1$

Assume they meet at some point. Eliminate y .

$$x^2 + 1 = 2x - 3 \Rightarrow x^2 - 2x + 4 = 0$$

$$(x - 1)^2 + 3 = 0, \text{ so } (x - 1)^2 = -3$$

a square cannot be negative: a contradiction

Your turn A3

Prove by contradiction that the curves $y = x^4 + 3$ and $y = 2x^2 + 1$ do not meet.



STRETCH

Worked example A4: infinitely many primes

Prove by contradiction that there are infinitely many prime numbers.

Assume the only primes are $p_1, \dots, p_n$, and let

$N = p_1 p_2 \cdots p_n + 1$. $N > 1$ has a prime factor q , so q is one of the p_i

q divides $p_1 \cdots p_n$ and N , so q divides 1

no prime divides 1: a contradiction

Your turn A4

Prove that there are infinitely many primes by assuming that $2, 3, \dots, p_n$ are the only primes, and considering

$$N = 2 \times 3 \times \cdots \times p_n + 1$$

Explain why N has a prime factor.

Worked example A5: cube roots

Prove by contradiction that $\sqrt[3]{2}$ is irrational.

An odd number cubed is odd, so if a^3 is even, a is even.

$\sqrt[3]{2} = \frac{a}{b}$, no common factors: $a^3 = 2b^3$, so $a = 2m$

$8m^3 = 2b^3$, so $b^3 = 4m^3$ and b is even

2 is a common factor: a contradiction

Your turn A5

Prove by contradiction that $\sqrt[3]{4}$ is irrational.

Worked example A6: multiply only by a positive

Prove by contradiction that $\tan \theta + \cot \theta \geq 2$

for $0 < \theta < \frac{\pi}{2}$

Assume it fails for some θ . As $\tan \theta > 0$, multiply by it.

$$\tan^2 \theta + 1 < 2 \tan \theta \Rightarrow (\tan \theta - 1)^2 < 0$$

a square cannot be negative: a contradiction

Your turn A6

Prove by contradiction that $e^x + 4e^{-x} \geq 4$ for all real x

Worked example A7: no stationary points

Prove by contradiction that the curve $y = x^3 + 6x + 4 \sin x$ has no stationary points.

Assume $\frac{dy}{dx} = 0$ for some value of x .

$$\frac{dy}{dx} = 3x^2 + 6 + 4 \cos x \geq 0 + 6 - 4$$

$\frac{dy}{dx} \geq 2$: a contradiction

Your turn A7

Prove by contradiction that the curve $y = x^3 - 3x^2 + 5x + 2$ has no stationary points.



B**Fluency**

Each question is a designed sequence. From one part to the next, ask yourself: what changed, and what did it do to the answer?

CORE*Secure the method*

B1 Write down the assumption you would make to begin a proof by contradiction of each statement.

(a) For all integers n , $n^2 + n$ is even.

(b) There are no integers a and b such that $3a + 6b = 1$

(c) $\sqrt{11}$ is irrational.

(d) For all integers n , if n^2 is odd, then n is odd.

[4 marks]

B2 n is an integer. Prove each statement by contradiction.

(a) If $7n + 2$ is odd, then n is odd.

(b) If n^3 is odd, then n is odd.

(c) If $n^2 + 2n$ is even, then n is even.

(d) If $5n^2 - 1$ is odd, then n is even.

[4 marks]

B3 Prove by contradiction that there are no integers a , b and c such that

(a) $6a + 9b = 2$

(b) $10a - 15b = 4$

(c) $4a + 6b = 9$

(d) $12a - 18b + 30c = 8$

[4 marks]

B4 Given that $\sqrt{2}$ is irrational, prove by contradiction that each number is irrational.

(a) $3\sqrt{2}$

(b) $5 + \sqrt{2}$

(c) $\frac{1}{\sqrt{2}}$

(d) $(1 + \sqrt{2})^2$

[4 marks]

B5 Prove by contradiction that each pair of graphs does not meet.

(a) $y = x^2 + 4x + 9$ and $y = 2x + 3$

(b) $y = 2x^2 - x + 4$ and $y = x^2 + 3x - 1$

(c) $y = x^4 + 7$ and $y = 4x^2 + 2$

[3 marks]



B6 Prove each inequality by contradiction.

(a) $x^2 + 9 \geq 6x$ for all real x

(b) $4x + \frac{1}{x} \geq 4$ for all $x > 0$

(c) $a^2 + b^2 \geq 2ab$ for all real a and b

(d) $x + \frac{4}{x-3} \geq 7$ for all $x > 3$

[4 marks]

STRETCH

B7 Prove by contradiction that each number is irrational.

(a) $\sqrt[3]{5}$

(b) $\sqrt[3]{9}$

(c) $\sqrt[4]{8}$

(d) $\sqrt[3]{6}$

[4 marks]

B8 Prove by contradiction that there are no integers a , b and c such that

(a) $a^2 - 4b = 2$

(b) $a^2 - 4b = 3$

(c) $a^2 + b^2 = 4c + 3$

(d) $a^3 = 4b + 2$

[4 marks]

B9 Prove each statement by contradiction.

(a) The curve $y = x^3 - 6x^2 + 15x$ has no stationary points.

(b) The curve $y = 5x + 3\sin x - 2\cos x$ has no stationary points.

(c) The curve $y = 2x + \ln(x^2 + 1)$ has no stationary points.

(d) The curves $y = e^{2x} + 3$ and $y = 2e^x$ do not meet.

[4 marks]

B10 Prove each inequality by contradiction.

(a) $\frac{x^2}{x^4 + 1} \leq \frac{1}{2}$ for all real x

(b) $\sin^4 x + \cos^4 x \geq \frac{1}{2}$ for all real x

(c) $\frac{x^2 + 5}{\sqrt{x^2 + 4}} \geq \frac{5}{2}$ for all real x

(d) $(a+b)\left(\frac{1}{a} + \frac{1}{b}\right) \geq 4$ for all positive a and b

[4 marks]



C

Reasoning

Show your working and explain your thinking. A correct answer without a reason does not score full marks.

CORE

Secure the method

- C1** To prove that “for all integers n , $n^2 - n$ is even”, Amir begins: “Assume that $n^2 - n$ is odd for all integers n .” Explain what is wrong with Amir’s assumption, and write a correct one. [2 marks]
- C2** To prove that the curves $y = x^4 + 5$ and $y = 3x^2$ do not meet, Bea writes: “Assume they meet. Then $x^4 - 3x^2 + 5 = 0$. My calculator gives no real solutions, so they do not meet.” Explain why Bea would score very few marks, and give the algebra she should have used. [2 marks]
- C3** Cal is asked to prove that “if n^2 is a multiple of 3, then n is a multiple of 3”. He writes: “ $9 = 3^2$, $36 = 6^2$ and $81 = 9^2$, so it is true.” Explain why this is not a proof, and give a correct proof by contradiction. [3 marks]
- C4** Dina proves that $x + \frac{1}{x} \geq 2$ for all $x > 0$. Her friend says the result also holds for all $x \neq 0$, and Dina replies that $x = 0$ is a counter-example. Explain why $x = 0$ is not a valid counter-example, give a valid one, and say which step of Dina’s proof needs $x > 0$ [3 marks]

STRETCH

- C5** Eli uses the method for $\sqrt{2}$ to “prove” that $\sqrt{9}$ is irrational: “ $a^2 = 9b^2$, so a^2 is a multiple of 9, so a is a multiple of 9, so $a = 9m$, ...” Find the step that is false, give a value of a that shows it, and explain why the method works for $\sqrt{5}$ but not for $\sqrt{9}$ [3 marks]
- C6** To show that $y = x^4 + 4x^2 + 3$ and $y = -x^2 - 1$ do not meet, Fay writes: “Assume they meet: $x^4 + 5x^2 + 4 = 0$. Since $x^2 > 0$ and $x^4 > 0$, the left side is greater than 4, so it cannot be 0” Identify the incorrect statement, correct it, and explain whether her conclusion still follows. [3 marks]
- C7** Gus says: “Euclid’s proof shows that $2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1$ must be prime.” Show that $2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1 = 59 \times 509$, and explain why Euclid’s proof is still valid. [3 marks]
- C8** (a) Prove that, for every integer n , n^2 leaves a remainder of 0 or 1 when it is divided by 3 (2)
- (b) Hence prove by contradiction that if $a^2 + b^2$ is a multiple of 3, then a and b are both multiples of 3 (3)
- [5 marks]



D**Problem solving**

You decide the method. Read the whole question first, and show every step.

CORE

Secure the method

- D1** Given that n is an integer, use algebra to prove by contradiction that
“if $n^3 + 5$ is odd, then n is even”
State your assumption clearly at the start, and finish with a conclusion. [3 marks]
- D2** (a) Use proof by contradiction to prove that $\sqrt{3}$ is irrational. You may use the fact that if a^2 is a multiple of 3, then a is a multiple of 3 (3)
- (b) Hence prove that $(2 + \sqrt{3})^2$ is irrational. (1)
- [4 marks]
- D3** *In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.*
The curve C has equation $y = x^2 + 2x + 7$ and the line l has equation $y = 4x + 5$
Use algebra to prove by contradiction that C and l do not meet. [3 marks]
- D4** A rectangle has an area of 36 cm^2 . One of its sides has length $x \text{ cm}$.
Use algebra to prove by contradiction that the perimeter of the rectangle is at least 24 cm , whatever the value of x [3 marks]



STRETCH

- D5** In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.

Figure 1 shows a sketch of part of the curve C with equation

$$y = x^3 - 3x^2 + 4x + \cos x$$

Near $x = 1$ the curve appears to have a stationary point.

- (a) Show that $\frac{dy}{dx} = 3(x-1)^2 + 1 - \sin x$

(1)

- (b) Use proof by contradiction to prove that C has no stationary points.

(3)

[4 marks]

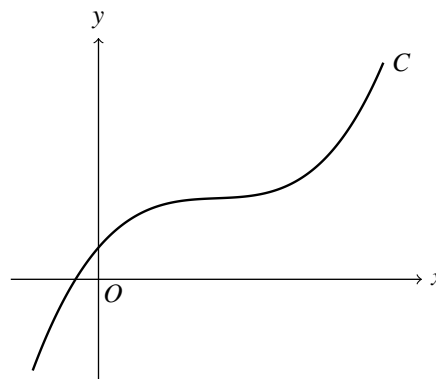


Figure 1

- D6** Use proof by contradiction to prove that the equation

$$x^3 + x + 1 = 0$$

has no rational solutions.

You may use the fact that any rational number can be written as $\frac{a}{b}$, where a and b are integers with no common factors and $b \neq 0$

[4 marks]

- D7** In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.

- (a) Show that $(\sin x + \cos x)^2 \equiv 1 + \sin 2x$

(1)

- (b) Hence use proof by contradiction to prove that, for all angles x with $180^\circ < x < 270^\circ$,
 $\sin x + \cos x < -1$

(3)

[4 marks]

- D8** (a) Prove that the product of two integers of the form $4k+1$, where k is an integer, is also of the form $4k+1$

(1)

- (b) By assuming that $p_1, p_2, \dots, p_r$ are the only primes of the form $4k+3$, and considering

$$N = 4p_1p_2 \cdots p_r - 1$$

prove by contradiction that there are infinitely many primes of the form $4k+3$

(4)

[5 marks]



E**Exam practice**

Exam-style questions in the style of WMA14. Write your answers in the spaces, and show every step.

CORE

Exam questions for everyone

- E1** *In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.*

The curve C_1 has equation $y = x^4 - 2x^3 + 3x^2$

The curve C_2 has equation $y = 2x^3 - 5x^2 + 4x - 3$

- (a)** Show that

$$x^4 - 4x^3 + 8x^2 - 4x + 3 \equiv (x^2 - 2x)^2 + (2x - 1)^2 + 2$$

(2)

- (b)** Hence use algebra to prove by contradiction that C_1 and C_2 do not intersect.

(3)

(Total for Question E1 is 5 marks)

- E2** **(a)** Given that $n \in \mathbb{Z}$, prove by contradiction that
“if n^3 is even, then n is even”

(2)

- (b)** Hence use proof by contradiction to prove that $\sqrt[3]{12}$ is irrational.

You may use the result of part (a) without proof.

(4)

(Total for Question E2 is 6 marks)



STRETCH

E3 In this question a , k and m are positive integers.

The k consecutive integers $a, a+1, a+2, \dots, a+k-1$ have sum S

(a) Show that $S = \frac{k(2a+k-1)}{2}$

(2)

(b) Explain why one of the numbers k and $2a+k-1$ is odd and the other is even.

(1)

(c) Hence prove by contradiction that, for $m \geq 1$, the number 2^m cannot be written as the sum of two or more consecutive positive integers.

(4)

(d) Write 40 as the sum of two or more consecutive positive integers.

(1)

(Total for Question E3 is 8 marks)

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