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A-LEVEL MATHEMATICS • FURTHER PURE 2

# First-Order Differential Equations

## Worked Solutions

Pearson Edexcel International A Level (WFM02) • Calculator allowed: show every step

Full worked solutions for the free First-Order Differential Equations worksheet, A-Level and International A-Level Maths


A **Examples** & Your turn page 2

B **Fluency** core skills page 2

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Tap a section to jump straight to it

**Full worked solutions** for every question on the Further Pure 2 First-Order Differential Equations worksheet, in the same order and with the same question numbers.

**How to mark your work.** Tick each step you got right, not just the final answer. At A-level most marks are for method, so a wrong final answer can still earn most of the marks. If you lost marks, find the *first* line where your working is different from ours. That is the thing to fix.

In Section E: **M** = method mark, **dM** = method mark that depends on the previous M mark, **A** = accuracy mark (needs the M mark before it), **B** = independent mark.

Question sheet | Exam questions by topic | Further Pure 2 exam questions | More worksheets


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# A Your turn answers

Compare each line of your working with the solution.

## CORE

**A1** IF $e^{-x}$: $ye^{-x} = \int e^{2x}\,dx$: $\boxed{y = \frac{1}{2}e^{3x} + ce^{x}}$

**A2** $\frac{dy}{dx} + \frac{3y}{x} = x$; IF $x^3$: $x^3y = \frac{x^5}{5} + c$: $\boxed{y = \frac{x^2}{5} + \frac{c}{x^3}}$

**A3** $y = e^{x} + ce^{-x}$; $3 = 1 + c$: $\boxed{y = e^{x} + 2e^{-x}}$

## STRETCH

**A4** IF $\sec x$: $y\sec x = \int \sec^2 x\,dx = \tan x + c$: $\boxed{y = \sin x + c\cos x}$

**A5** $\frac{dz}{dx} = 3y^2\frac{dy}{dx}$, so $\frac{dz}{dx} - z = e^{2x}$; IF $e^{-x}$: $\boxed{y^3 = e^{2x} + ce^{x}}$

**A6** $\frac{dy}{dx} - \frac{2y}{x} = x^2$; IF $\frac{1}{x^2}$: $y = x^3 + cx^2$; $4 = 1 + c$: $\boxed{y = x^3 + 3x^2}$

**A7** IF $e^{t/10}$: $Pe^{t/10} = 500e^{t/10} + c$, $c = -500$: $\boxed{P = 500\left(1 - e^{-t/10}\right)}$

# B Fluency solutions

One mark for each correct part. The notes in italics explain why each sequence was chosen.

## CORE

**B1** **(a)** $\boxed{e^{4x}}$ **(b)** $\boxed{e^{x^2}}$ **(c)** $\boxed{x}$ **(d)** $e^{-\int \tan x} = e^{\ln\cos x}$: $\boxed{\cos x}$

$e^{\ln f(x)} = f(x)$

**B2** **(a)** $\boxed{y = 2 + ce^{-4x}}$ **(b)** $\boxed{y = \frac{1}{2} + ce^{-x^2}}$ **(c)** $\boxed{y = \frac{x}{2} + \frac{c}{x}}$ **(d)** $\boxed{y = \tan x + c\sec x}$

*Divide every term, including the constant, by the integrating factor.*

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**B3** **(a)** IF $e^{3x}$, by parts: $y = 2x - \frac{2}{3} + ce^{-3x}$ **(b)** $c = \frac{2}{3}$ **(c)** $c = 0$: **the line** $y = 2x - \frac{2}{3}$

**(d)** $y \approx 2x - \frac{2}{3}$ **for every solution**

*The exponential term dies away, so all curves approach one line.*

**B4** **(a)** $y = \frac{x^2 + 2c}{2(x+1)}$ **(b)** $y = \frac{x + c}{x^2}$ **(c)** $\frac{d}{dx}\left(\frac{y}{\cos x}\right) = 1$: $y = (x + c)\cos x$ **(d)** $y = (x^2 + c)e^{-x}$

*Some equations are already exact: no integrating factor is needed.*

**B5** **(a)** $\frac{dz}{dx} = 2y\frac{dy}{dx}$ **(b)** multiply by 2: $2y\frac{dy}{dx} - 2y^2 = 2x$, so $\frac{dz}{dx} - 2z = 2x$

**(c)** IF $e^{-2x}$, by parts: $y^2 = ce^{2x} - x - \frac{1}{2}$

*The chain rule links $z$ and $y$*

**B6** **(a)** $y = x + \frac{c}{x}$ **(b)** $y = x$ **(c)** $y = x + \frac{2}{x}$ **(d)** $x = 0$ **and** $y = x$

*One constant gives a whole family of curves.*

## STRETCH

**B7** **(a)** $e^{-x}$ **(b)** $\frac{d}{dx}\left(ye^{-x}\right) = x$: $y = \left(\frac{x^2}{2} + c\right)e^{x}$ **(c)** $c = 1$ **(d)** $3e^2$

*The integrating factor can cancel an exponential on the right.*

**B8** **(a)** $\frac{du}{dx} = 1 + \frac{dy}{dx} = \frac{1}{u}$ **(b)** $u\,du = dx$: $u^2 = 2x + c$ **(c)** $u(0) = 2$: $c = 4$ **(d)** $y = \sqrt{2x + 4} - x$

*The substitution makes the equation separable.*

**B9** **(a)** divide by $-y^2$: $-\frac{1}{y^2}\frac{dy}{dx} + \frac{2}{xy} = x$, and $\frac{dz}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$ **(b)** IF $x^2$: $z = \frac{x^2}{4} + \frac{c}{x^2}$

**(c)** $y = \frac{4x^2}{x^4 + 4c}$ **(d)** $1 = \frac{4}{1 + 4c}$: $y = \frac{4x^2}{x^4 + 3}$

*Reciprocate the whole expression for $z$*

**B10** **(a)** $e^{t/50}$ **(b)** $S = 100\left(1 - e^{-t/50}\right)$ **(c)** **100** **(d)** $e^{-t/50} = \frac{1}{2}$: $t = 50\ln 2$

*The long-term value is the equilibrium of the equation.*

# C Reasoning solutions

Your wording can be different. Check that your reason makes the same point.

## CORE

**C1** The right side must also be multiplied: $\boxed{e^{2x}\frac{dy}{dx}+2e^{2x}y=xe^{2x}}$

**C2** The constant must also be divided by $e^{2x}$: $\boxed{y=xe^{-2x}+ce^{-2x}}$

**C3** Divide by $x$ first: $\frac{dy}{dx}+\frac{3y}{x}=1$, IF $x^3$; $x^3y=\frac{x^4}{4}+c$: $\boxed{y=\frac{x}{4}+\frac{c}{x^3}}$

**C4** $P(x)=-\tan x$, so the IF is $e^{-\int \tan x\,dx}=e^{\ln\cos x}$: $\boxed{\cos x}$

## STRETCH

**C5** $\frac{1}{x^2+c}\neq\frac{1}{x^2}+\frac{1}{c}$; reciprocate the whole expression: $\boxed{y=\frac{1}{x^2+c}}$

**C6** $e^{2\ln x}=e^{\ln x^2}=x^2$; $x^2y=\frac{x^4}{4}+c$: $\boxed{y=\frac{x^2}{4}+\frac{c}{x^2}}$

**C7** As $t\to\infty$ his $T\to 0$, not the room temperature 20; $T-20$ decays: $\boxed{T=20+Ae^{-kt}}$

**C8** **(a)** $\frac{d}{dx}(Iy)=I\frac{dy}{dx}+yI'$ and $I'=PI$, so $\frac{d}{dx}(Iy)=I\left(\frac{dy}{dx}+Py\right)$, as required

**(b)** $ye^{ax}=\frac{b}{a}e^{ax}+c$, so $y=\frac{b}{a}+ce^{-ax}\to\frac{b}{a}$ as $x\to\infty$, as required

# D Problem solving solutions

There is often more than one correct method. Any complete, correct method scores full marks.

## CORE

**D1** **(a)** $e^{\int 2\tan x\,dx} = e^{2\ln\sec x} = \sec^2 x$, as required

**(b)** $y\sec^2 x = \int \sec x\tan x\,dx = \sec x + c$; $3 = 1 + c$: $\boxed{y = \cos x + 2\cos^2 x}$

**D2** IF $x^3$: $x^3 y = \int xe^x\,dx = (x-1)e^x + c$; $2 = 0 + c$: $\boxed{y = \dfrac{(x-1)e^x + 2}{x^3}}$

**D3** $\dfrac{dz}{dx} = 3y^2\dfrac{dy}{dx}$; multiply (I) by 3: $3y^2\dfrac{dy}{dx} + \dfrac{3y^3}{x} = 3x$, so $\dfrac{dz}{dx} + \dfrac{3z}{x} = 3x$, as required

**D4** IF $\dfrac{1}{x}$: $\dfrac{y}{x} = \int \cos x\,dx = \sin x + c$; $y = x\sin x + cx$; $\pi = \dfrac{\pi}{2} + \dfrac{c\pi}{2}$, $c = 1$: $\boxed{y = x\sin x + x}$

## STRETCH

**D5** **(a)** $\dfrac{dz}{dx} = -2y^{-3}\dfrac{dy}{dx}$; multiply by $-2y^{-3}$: $-2y^{-3}\dfrac{dy}{dx} + 2y^{-2} = -2e^x$, as required

**(b)** IF $e^{2x}$: $ze^{2x} = -\dfrac{2}{3}e^{3x} + c$: $\boxed{y^2 = \dfrac{1}{ce^{-2x} - \dfrac{2}{3}e^x}}$

**D6** **(a)** $\dfrac{dz}{dx} = e^y\dfrac{dy}{dx}$, so $\dfrac{dz}{dx} + z = x$, as required

**(b)** $z = x - 1 + ce^{-x}$; $z(0) = 1$ gives $c = 2$: $\boxed{y = \ln\left(x - 1 + 2e^{-x}\right)}$

**D7** **(a)** $\dfrac{dP}{dt} - 0.1P = -20$, IF $e^{-0.1t}$: $P = 200 + Ae^{0.1t}$, $A = -50$: $\boxed{P = 200 - 50e^{0.1t}}$

**(b)** $e^{0.1t} = 4$: $\boxed{t = 10\ln 4}$ years; after this $P$ would be negative, so the model fails

**D8** **(a)** IF $x^2$: $x^2 y = \dfrac{3}{2}x^4 + c$: $\boxed{y = \dfrac{3}{2}x^2 + \dfrac{c}{x^2}}$

**(b)** $2 = \dfrac{3}{2} + c$: $\boxed{y = \dfrac{3}{2}x^2 + \dfrac{1}{2x^2}}$

**(c)** $\dfrac{dy}{dx} = 3x - x^{-3} = 0$, $x^4 = \dfrac{1}{3}$; $\dfrac{d^2y}{dx^2} = 3 + 3x^{-4} > 0$: $\boxed{\left(3^{-\frac{1}{4}},\ \sqrt{3}\right)}$

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# E Exam practice solutions

Mark-scheme notes are shown on the right of each line.

## CORE *Exam questions for everyone*

**E1** **(a)** IF $= e^{\int \cot x \, dx}$ *M1*

$= e^{\ln \sin x} = \sin x$, as required *A1**

**(b)** $\frac{d}{dx}(y \sin x) = 2 \sin x \cos x = \sin 2x$ *M1*

$\boxed{y = \frac{c - \frac{1}{2}\cos 2x}{\sin x}}$ *A1*

**(c)** $1 = c + \frac{1}{2}$, so $c = \frac{1}{2}$ *M1*

$y = \frac{\frac{1}{2}(1 - \cos 2x)}{\sin x} = \frac{\sin^2 x}{\sin x}$: $\boxed{y = \sin x}$ *A1*

**E2** **(a)** $\frac{dz}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$ *B1*

divide (I) by $xy^2$: $\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = x$ *M1*

$-\frac{dz}{dx} + \frac{z}{x} = x$, so $\frac{dz}{dx} - \frac{z}{x} = -x$, as required *A1**

**(b)** IF $\frac{1}{x}$: $\frac{d}{dx}\left(\frac{z}{x}\right) = -1$ *M1*

$z = -x^2 + cx$ *A1*

$\boxed{y = \frac{1}{cx - x^2}}$ *A1*

**(c)** $\frac{1}{2} = \frac{1}{c - 1}$: $\boxed{y = \frac{1}{3x - x^2}}$ *B1*

**STRETCH**

**E3** **(a)** $\frac{dz}{dx} = 2y\frac{dy}{dx}$ *B1*

(I) becomes $\frac{dz}{dx} + z\tan x = \sec x$, as required *M1 A1**

**(b)** IF $= e^{\int \tan x\,dx} = \sec x$ *M1 A1*

$z\sec x = \int \sec^2 x\,dx = \tan x + c$ *M1*

$\boxed{y^2 = \sin x + c\cos x}$ *A1*

**(c)** $4 = 0 + c$, so $y^2 = \sin x + 4\cos x$ *M1*

$y^2 = \frac{\sqrt{2}}{2}(1+4) = \frac{5\sqrt{2}}{2}$: $\boxed{y = \sqrt{\frac{5\sqrt{2}}{2}}}$ *A1*