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INTERNATIONAL A LEVEL MATHEMATICS • PURE 4 (WMA14)

Pure 4 Practice Paper 3

Worked Solutions

AIMING FOR A/A*

Written in the style of the Pearson Edexcel International Advanced Level paper (WMA14/01) • 75 marks

Every question solved in full on the paper itself, the way your teacher would write it out

How to read these solutions. Each page is the real question page, with a full answer written on the answer lines.

Red pen

M1 A1

boxed

the working a full-marks answer shows, with a short note on why each step is taken

the mark earned on that line: M = method, A = accuracy, B = independent mark

the final answer to each part

Sit the paper first. These solutions are most useful after you have tried every question yourself under timed conditions. Mark your paper with the mark scheme, then use this booklet to see exactly how each lost mark could have been earned.

Q	Topic	Marks	Page
1	The general term of a series	9	2
2	Parametric area and volume	10	3
3	A reaction modelled by a differential equation	10	5
4	An equilateral triangle on a line	9	7
5	Proofs about rational roots	9	8
6	Tangents to the folium $x^3 + y^3 = 3xy$	9	9
7	An equiangular spiral	9	11
8	Filling a leaking container	10	13

Question paper & mark scheme | Exam questions by topic | More practice papers

1. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

$$f(x) = \frac{3-x}{(1-x)^2(2-x)}$$

(a) Find, in its simplest form, the coefficient of x^n in the series expansion of $f(x)$ in ascending powers of x.

(5)

(b) State the range of values of x for which the expansion is valid.

(1)

(c) Hence find the exact value of

$$\sum_{n=0}^{\infty} \frac{2n+1}{2^n}$$

(3)

(a) $f(x) \equiv \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{2-x}$ (repeated factor needs two fractions) M1

$3 - x \equiv A(1-x)(2-x) + B(2-x) + C(1-x)^2$

$x = 1$: $2 = B$ $x = 2$: $1 = C$ $x = 0$: $3 = 2A + 4 + 1 \Rightarrow A = -1$ A1

So $f(x) = -(1-x)^{-1} + 2(1-x)^{-2} + (2-x)^{-1}$

$(1-x)^{-1} = 1 + x + x^2 + \ldots$ has x^n coefficient 1

$(1-x)^{-2} = 1 + 2x + 3x^2 + \ldots$ has x^n coefficient $(n+1)$ M1

$(2-x)^{-1} = \frac{1}{2}(1-x/2)^{-1} = \frac{1}{2} + x/4 + x^2/8 + \ldots$ has x^n coefficient $1/2^{n+1}$ M1

Coefficient $= -1 + 2(n+1) + 1/2^{n+1}$ $\boxed{2n + 1 + 1/2^{n+1}}$ A1

(b) $(2-x)^{-1}$ needs $|x| < 2$, but $(1-x)^{-1}$ needs $|x| < 1$ $\boxed{|x| < 1}$ B1

(c) $x = \frac{1}{2}$ is inside $|x| < 1$, so the series for $f(\frac{1}{2})$ converges ✓

$f(\frac{1}{2}) = (5/2) \div (3/8) = \frac{20}{3}$ and $f(\frac{1}{2}) = \sum (2n+1)/2^n + \sum (1/2^{n+1})(1/2^n)$ M1

$\sum (1/2^{n+1})(1/2^n) = \sum \frac{1}{2}(\frac{1}{4})^n = \frac{1}{2} \div (1 - \frac{1}{4}) = \frac{2}{3}$ (geometric series) M1

$\sum (2n+1)/2^n = \frac{20}{3} - \frac{2}{3}$ $\boxed{6}$ A1

(Total for Question 1 is 9 marks)

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2. In this question you must show all stages of your working.

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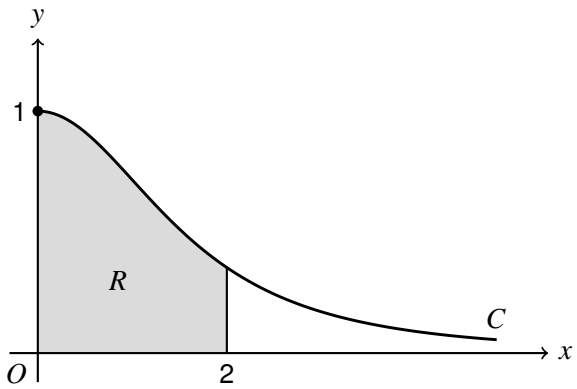

Figure 1

Figure 1 shows the curve C with parametric equations

$$x = 2\tan\theta, \qquad y = \cos^3\theta \qquad 0 \leqslant \theta < \frac{\pi}{2}$$

The region R, shown shaded in Figure 1, is bounded by C, the coordinate axes and the line $x = 2$

(a) Find the exact area of R.

(4)

The region R is rotated through 2π radians about the x-axis to form a solid of revolution.

(b) Find the exact volume of the solid.

(6)

(a) Area $= \int y \frac{dx}{d\theta}\, d\theta$, $\frac{dx}{d\theta} = 2\sec^2\theta$ M1

$= \int \cos^3\theta \times 2\sec^2\theta \, d\theta = \int 2\cos\theta \, d\theta$ A1

$x = 0 \Rightarrow \theta = 0$; $x = 2 \Rightarrow \tan\theta = 1 \Rightarrow \theta = \pi/4$ [2sinθ] from 0 to π/4 M1

$= 2 \times \sqrt{2}/2 - 0$ Area $= \sqrt{2}$ A1

(b) $V = \pi \int y^2 \frac{dx}{d\theta}\, d\theta = \pi \int \cos^6\theta \times 2\sec^2\theta \, d\theta = 2\pi \int \cos^4\theta \, d\theta$ M1

$\cos^4\theta = (\cos^2\theta)^2 = \frac{1}{4}(1 + \cos 2\theta)^2 = \frac{1}{4}(1 + 2\cos 2\theta + \cos^2 2\theta)$ M1

$\cos^2 2\theta = \frac{1}{2}(1 + \cos 4\theta) \Rightarrow \cos^4\theta = 3/8 + \frac{1}{2}\cos 2\theta + 1/8 \cos 4\theta$ M1

$\int \cos^4\theta \, d\theta = 3\theta/8 + \frac{1}{4}\sin 2\theta + \sin 4\theta/32$ A1

From 0 to π/4: $3\pi/32 + \frac{1}{4}\sin(\pi/2) + \sin(\pi)/32 - 0 = 3\pi/32 + \frac{1}{4}$

$V = 2\pi(3\pi/32 + \frac{1}{4})$ dM1

$V = \pi/2 + 3\pi^2/16$ A1

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Question 2 continued

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(Total for Question 2 is 10 marks)

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3. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

In a chemical reaction, x grams of a product have formed t minutes after the reaction starts.

The reaction is modelled by the differential equation

$$\frac{dx}{dt} = \frac{(4-x)(2-x)}{8} \qquad 0 \leqslant x < 2$$

At the start of the reaction no product has formed.

(a) Find x in terms of t.

(6)

(b) Sketch the graph of x against t, stating the equation of its asymptote.

(2)

(c) Find the exact time at which 1 gram of the product has formed.

(2)

(a) Separate: $\int \frac{8}{(4-x)(2-x)}\,dx = \int 1\,dt$ M1

$8 \equiv A(2-x) + B(4-x)$: $x = 2 \Rightarrow B = 4$, $x = 4 \Rightarrow A = -4$

So $8/((4-x)(2-x)) = 4/(2-x) - 4/(4-x)$ B1

$\int \frac{4}{2-x}\,dx = -4\ln(2-x)$, $\int \frac{4}{4-x}\,dx = -4\ln(4-x)$ (watch the minus signs)

$-4\ln(2-x) + 4\ln(4-x) = t + c$ M1

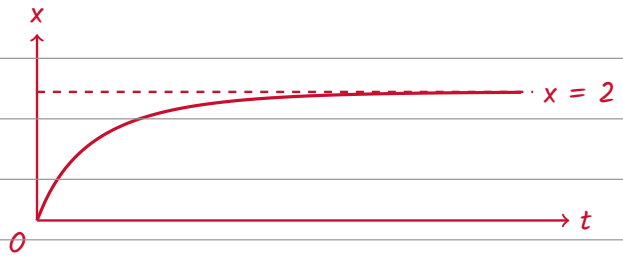
$4\ln((4-x)/(2-x)) = t + c$ A1

$t = 0, x = 0$: $4\ln 2 = c \Rightarrow \ln((4-x)/(2-x)) = t/4 + \ln 2 \Rightarrow (4-x)/(2-x) = 2e^{t/4}$ M1

$4 - x = 4e^{t/4} - 2xe^{t/4} \Rightarrow x(2e^{t/4} - 1) = 4e^{t/4} - 4$

$\boxed{x = 4(e^{t/4} - 1)/(2e^{t/4} - 1)}$ A1

(b) As $t \to \infty$, $x \to 4e^{t/4}/2e^{t/4} = 2$. The model needs $x < 2$, so not 4.



B1

Increasing from 0, levelling off $\boxed{\text{asymptote } x = 2}$ B1

(c) $x = 1$: $(4-1)/(2-1) = 3 = 2e^{t/4} \Rightarrow e^{t/4} = 3/2$ M1

$\boxed{t = 4\ln(3/2) \text{ minutes}}$ A1

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Question 3 continued

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(Total for Question 3 is 10 marks)

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4. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

The line l has equation

$$\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

The point A has coordinates (3, 2, 5).

The points P and Q lie on l and the triangle APQ is equilateral.

(a) Find the coordinates of P and of Q.

(7)

(b) Find the exact area of the triangle APQ.

(2)

(a) Find the foot F of the perpendicular from A first: $F = (1 + 2\lambda, -2 + \lambda, 3 - \lambda)$

$AF = (2\lambda - 2, \lambda - 4, -\lambda - 2)$, $AF \cdot (2, 1, -1) = 4\lambda - 4 + \lambda - 4 + \lambda + 2 = 6\lambda - 6 = 0$ M1

$\lambda = 1 \Rightarrow F(3, -1, 2)$ A1

$AF = (0, -3, -3) \Rightarrow |AF| = \sqrt{0 + 9 + 9} = 3\sqrt{2}$ M1

APQ equilateral: AF is a line of symmetry, so F is the midpoint of PQ and angle FAP = 30°

$FP = AF \tan 30° = 3\sqrt{2} \div \sqrt{3} = \sqrt{6}$ M1

$|d| = \sqrt{4 + 1 + 1} = \sqrt{6}$, so P and Q are one step of d from F: $\lambda = 1 \pm 1$ M1

$\lambda = 0$: $\boxed{P(1, -2, 3)}$ A1

$\lambda = 2$: $\boxed{Q(5, 0, 1)}$ A1

Check: $AP = (-2, -4, -2)$, $AQ = (2, -2, -4)$, $PQ = (4, 2, -2)$: all have length $\sqrt{24}$ ✓

(b) Side $= 2\sqrt{6}$: Area $= \frac{1}{2} \times PQ \times AF = \frac{1}{2} \times 2\sqrt{6} \times 3\sqrt{2} = 3\sqrt{12}$ M1

$\boxed{\text{Area} = 6\sqrt{3}}$ A1

(Total for Question 4 is 9 marks)

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5. In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

(a) Given that a, b and c are odd integers, prove by contradiction that the equation

$$ax^2 + bx + c = 0$$

has no rational solutions.

(6)

(b) The equation $x^3 + x + 1 = 0$ has exactly one real root α.

Prove by contradiction that α is irrational.

(3)

(a) Assume there is a rational solution $x = p/q$, where p and q are integers with no common factor ($q \neq 0$). So p and q are not both even. B1

$a(p/q)^2 + b(p/q) + c = 0$; multiply by q^2: $ap^2 + bpq + cq^2 = 0$ M1

a, b, c are odd. Look at the three possible cases for p and q: M1

p odd, q odd: ap^2, bpq, cq^2 are all odd: odd + odd + odd = odd A1

p even, q odd: ap^2 even, bpq even, cq^2 odd: even + even + odd = odd A1

p odd, q even: ap^2 odd, bpq even, cq^2 even: odd + even + even = odd

In every case $ap^2 + bpq + cq^2$ is odd, so it cannot equal 0. This contradicts the assumption, so $ax^2 + bx + c = 0$ has no rational solutions. A1

(b) Assume $\alpha = p/q$, integers with no common factor (not both even):

$p^3/q^3 + p/q + 1 = 0$; multiply by q^3: $p^3 + pq^2 + q^3 = 0$ B1

Same three cases for p and q: M1

both odd: odd + odd + odd = odd; p even, q odd: even + even + odd = odd;

p odd, q even: odd + even + even = odd. Odd $\neq 0$ in every case:

a contradiction, so α is irrational. A1

(Total for Question 5 is 9 marks)

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6. In this question you must show all stages of your working.

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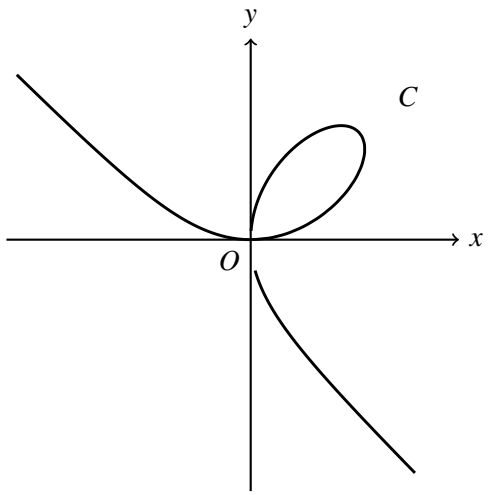

Figure 2

Figure 2 shows the curve C with equation

$$x^3 + y^3 = 3xy$$

(a) Find $\frac{dy}{dx}$ in terms of x and y.

(3)

(b) Find the exact coordinates of the point on C, other than O, at which the tangent to C is parallel to the x-axis.

(4)

(c) Hence write down the exact coordinates of the point on C, other than O, at which the tangent to C is parallel to the y-axis, giving a reason for your answer.

(2)

(a) Differentiate implicitly: $3x^2 + 3y^2 \frac{dy}{dx} = \ldots$ M1

$\ldots = 3y + 3x\, dy/dx$ (product rule on $3xy$) M1

$(dy/dx)(3y^2 - 3x) = 3y - 3x^2$ $\quad dy/dx = (y - x^2) / (y^2 - x)$ A1

(b) Parallel to the x-axis: $dy/dx = 0 \Rightarrow y - x^2 = 0 \Rightarrow y = x^2$ M1

Substitute in the curve: $x^3 + x^6 = 3x \cdot x^2 = 3x^3$

$x^6 - 2x^3 = 0 \Rightarrow x^3(x^3 - 2) = 0$ M1

$x = 0$ gives 0, but there $y^2 - x = 0$ too, so dy/dx is not defined: reject it.

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Question 6 continued

$x^3 = 2 \Rightarrow x = 2^{1/3}$, $y = x^2 = 2^{2/3}$ (and $y^2 - x = 2^{4/3} - 2^{1/3} \neq 0$ ✓) A1

$(2^{1/3}, 2^{2/3})$ A1

(c) Swapping x and y leaves $x^3 + y^3 = 3xy$ unchanged, so C is symmetrical

in the line $y = x$. Reflecting in $y = x$ turns a horizontal tangent into a vertical one. B1

$(2^{2/3}, 2^{1/3})$ B1

(Total for Question 6 is 9 marks)

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7. In this question you must show all stages of your working.

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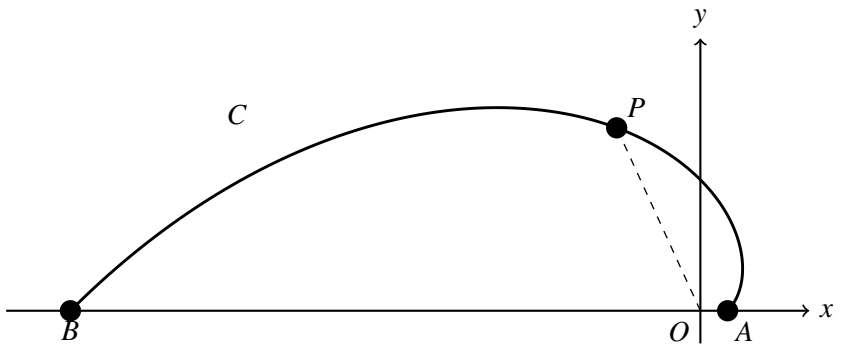

Figure 3

Figure 3 shows the curve C with parametric equations

$$x = e^t \cos t, \qquad y = e^t \sin t \qquad 0 \leqslant t \leqslant \pi$$

The curve starts at A, where $t = 0$, and ends at B, where $t = \pi$

The point P on C has parameter t.

(a) Find $\dfrac{dy}{dx}$ in terms of t.

(2)

(b) Prove that, for every point P on C, the angle between the tangent to C at P and the line OP is 45°

(4)

(c) Find the exact coordinates of the point on C at which the tangent is parallel to the x-axis.

(3)

(a) $\frac{dx}{dt} = e^t\cos t - e^t\sin t = e^t(\cos t - \sin t)$, $\frac{dy}{dt} = e^t(\sin t + \cos t)$ M1

$dy/dx = (\sin t + \cos t) / (\cos t - \sin t)$ A1

(b) Tangent direction $(\cos t - \sin t, \sin t + \cos t)$ (divide dx/dt and dy/dt by e^t)

OP direction $(\cos t, \sin t)$ ($OP = e^t(\cos t, \sin t)$) M1

Scalar product $= \cos^2 t - \sin t \cos t + \sin^2 t + \sin t \cos t = 1$ M1

Lengths: $\sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2} = \sqrt{2\cos^2 t + 2\sin^2 t} = \sqrt{2}$, and 1 M1

$\cos\phi = 1/\sqrt{2}$ for every t, so the angle is 45° at every point P on C ✓ A1

(c) Parallel to the x-axis: $dy/dt = 0 \Rightarrow \sin t + \cos t = 0 \Rightarrow \tan t = -1$ M1

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Question 7 continued

The calculator gives $-\pi/4$, which is outside $0 \leqslant t \leqslant \pi$. So $t = 3\pi/4$ $(\cos t - \sin t \neq 0)$ A1

$x = e^{3\pi/4}\cos(3\pi/4) = -(\sqrt{2}/2)e^{3\pi/4}$, $y = e^{3\pi/4}\sin(3\pi/4) = (\sqrt{2}/2)e^{3\pi/4}$

$(-(\sqrt{2}/2)e^{3\pi/4}, (\sqrt{2}/2)e^{3\pi/4})$ A1

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(Total for Question 7 is 9 marks)

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8. In this question you must show all stages of your working.

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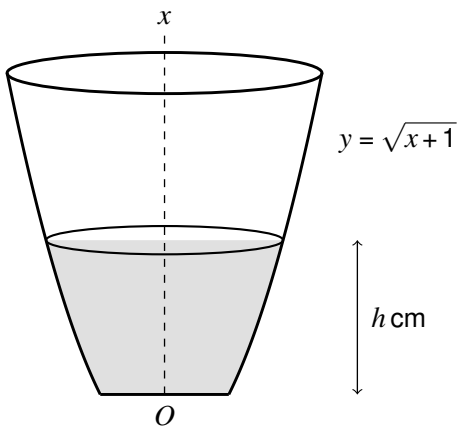

Figure 4

A container is formed by rotating the curve with equation $y = \sqrt{x+1}$, $0 \leqslant x \leqslant 5$, through 2π radians about the x-axis and closing the end at $x = 0$

The container stands on this flat end with the x-axis vertical, as shown in Figure 4. Lengths are in centimetres.

When the depth of water in the container is h cm, the volume of water is V cm^3.

(a) Show that $V = \pi\left(\frac{1}{2}h^2 + h\right)$

(2)

The container is initially empty. Water is poured in at a constant rate of 4π cm^3 s^{-1}, and at the same time water leaks out at a rate of πh cm^3 s^{-1}.

(b) Find the time t seconds, in terms of h, for the depth of the water to reach h cm.

(6)

(c) Find the time taken for the depth to reach 3 cm, and explain why, according to the model, the depth never reaches 4 cm.

(2)

(a) Water of depth h fills the solid from x = 0 to x = h:

$V = \pi\int y^2\,dx = \pi\int (x + 1)\,dx$, from 0 to h — M1

$= \pi[\frac{1}{2}x^2 + x]$ from 0 to h $= \pi(\frac{1}{2}h^2 + h)$ ✓ — A1*

(b) Rate in − rate out: $dV/dt = 4\pi - \pi h$ — M1

$dV/dh = \pi(h + 1)$, so $dV/dt = \pi(h + 1)\,dh/dt \Rightarrow \pi(h + 1)\,dh/dt = \pi(4 - h)$ — M1

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Question 8 continued

Separate: $\int (h + 1)/(4 - h)\, dh = \int 1\, dt$ M1

$(h + 1)/(4 - h) = (5 - (4 - h))/(4 - h) = -1 + 5/(4 - h) \Rightarrow \int \ldots dh = -h - 5\ln(4 - h)$ M1

$t = -h - 5\ln(4 - h) + c$ A1

$t = 0, h = 0$: $0 = -5\ln 4 + c \Rightarrow c = 5\ln 4$ $\boxed{t = 5\ln(4/(4 - h)) - h}$ A1

(c) $h = 3$: $t = 5\ln 4 - 3 = 6.931\ldots - 3$ $\boxed{t = 3.93 \text{ seconds}}$ B1

As $h \to 4$, $4/(4 - h) \to \infty$, so $\ln(4/(4 - h)) \to \infty$ and $t \to \infty$.

At $h = 4$, water leaks out (4π) as fast as it pours in, so the depth stops rising. B1

(Total for Question 8 is 10 marks)

TOTAL FOR PAPER IS 75 MARKS

END

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