Functions and Graphs

Worked Solutions

Suitable for:

Pearson Edexcel International A-Level (iAL) — Pure Mathematics 3

Question 1 — Worked Solution

(a) Evaluate f(1), f(3), f(4)

$$f(1) = |2(1) - 5| = |-3| = 3$$

$$f(3) = |2(3) - 5| = |1| = 1$$

$$f(4) = |2(4) - 5| = |3| = 3$$

(b) Solve |2x – 5| = 3

Case 1: $2x - 5 = 3 \Rightarrow 2x = 8 \Rightarrow x = 4$

Case 2: $2x - 5 = -3 \Rightarrow 2x = 2 \Rightarrow x = 1$

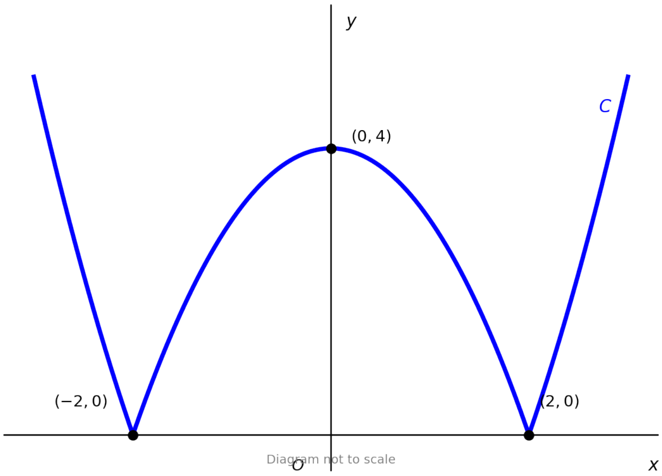
$$\therefore x = 1 \quad \text{or} \quad x = 4$$

Question 2 — Worked Solution

Sketch $y = |x^2 - 4|$

$x^2 - 4 = 0$ at $x = \pm 2$. The parabola is below the x-axis for $-2 < x < 2$, so this portion is reflected.

Key points: $(\pm 2, 0)$, and at $x = 0$: $|0 - 4| = 4$, giving $(0, 4)$.



Curve touches x-axis at $(-2, 0)$ and $(2, 0)$; y-intercept at $(0, 4)$.

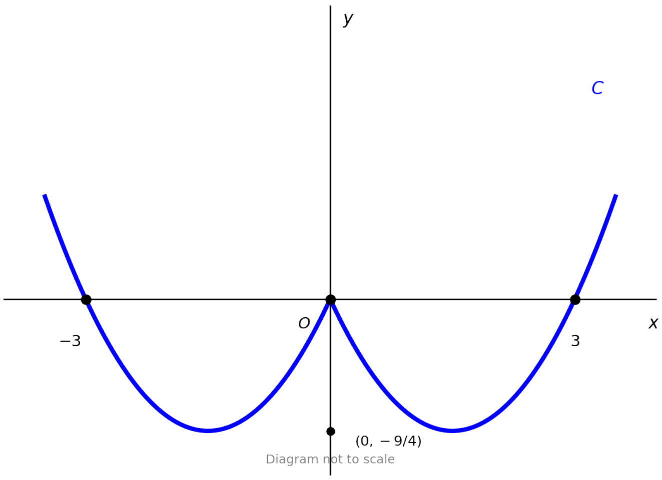
Question 3 — Worked Solution

(a) Sketch y = f(|x|)

$f(|x|) = |x|^2 - 3|x| = x^2 - 3|x|$.

For $x \geq 0$: $y = x^2 - 3x$ (same as f(x)).

For $x < 0$: $y = x^2 + 3x$ (reflection of $x \geq 0$ portion in the y-axis). The graph is even (symmetric about y-axis).



(b) Roots of f(|x|) = 0

$f(|x|) = 0$ when $|x|(|x| - 3) = 0$, so $|x| = 0$ or $|x| = 3$.

$$\therefore x = 0, \quad x = 3, \quad x = -3$$

Question 4 — Worked Solution

(a) Range of f

At $x = 2$ (minimum of domain): $f(2) = \sqrt{0} + 3 = 3$. f is increasing for $x \geq 2$.

Range: $f(x) \geq 3$

(b) One-to-one

f is strictly increasing on its domain $x \geq 2$ (as $\sqrt{(x-2)}$ is strictly increasing). Therefore each value in the range corresponds to exactly one value in the domain.

f is one-to-one since it is strictly increasing on its domain.

(c) Inverse function

Let $y = \sqrt{(x-2)} + 3$. Rearrange for x:

$$y - 3 = \sqrt{x-2} \;\Rightarrow\; (y-3)^2 = x - 2 \;\Rightarrow\; x = (y-3)^2 + 2$$

$$\therefore f^{-1}(x) = (x-3)^2 + 2$$

Domain: $x \geq 3$ (= range of f). Range: $f^{-1}(x) \geq 2$ (= domain of f).

Question 5 — Worked Solution

(a) fg(x) and gf(x)

$$fg(x) = f(g(x)) = f(x^2 - 3) = 2(x^2 - 3) + 1 = 2x^2 - 6 + 1$$

$$\therefore fg(x) = 2x^2 - 5$$

$$gf(x) = g(f(x)) = g(2x + 1) = (2x + 1)^2 - 3 = 4x^2 + 4x + 1 - 3$$

$$\therefore gf(x) = 4x^2 + 4x - 2$$

(b) Solve fg(x) = 27

$$2x^2 - 5 = 27 \Rightarrow 2x^2 = 32 \Rightarrow x^2 = 16$$

$$\therefore x = 4 \quad \text{or} \quad x = -4$$

Question 6 — Worked Solution

(a) Find ff(x)

$$ff(x) = f\left(\frac{1}{x-2}\right) = \frac{1}{\frac{1}{x-2} - 2}$$

Simplify the denominator:

$$\frac{1}{x-2} - 2 = \frac{1 - 2(x-2)}{x-2} = \frac{1 - 2x + 4}{x-2} = \frac{5 - 2x}{x-2}$$

So:

$$ff(x) = \frac{1}{\frac{5-2x}{x-2}} = \frac{x-2}{5-2x}$$

$$\therefore ff(x) = \frac{x-2}{5-2x}$$

(b) Domain of ff(x)

f(x) requires x ≠ 2. Also ff(x) requires f(x) ≠ 2, i.e. 1/(x–2) ≠ 2, so x ≠ 5/2.

Domain: x ∈ ■, x ≠ 2, x ≠ 5/2

Question 7 — Worked Solution

(a) Find $f^{-1}(x)$

Let y = 3x – 2. Rearrange for x:

$$y + 2 = 3x \Rightarrow x = \frac{y+2}{3}$$

$$\therefore f^{-1}(x) = \frac{x+2}{3}$$

(b) Domain of $f^{-1}(x)$

The range of f(x) = 3x – 2 is ■, so the domain of f^{-1} is also ■.

Domain: x ∈ ■

(c) Solve $f(x) = f^{-1}(x)$

$$3x - 2 = \frac{x+2}{3} \Rightarrow 9x - 6 = x + 2 \Rightarrow 8x = 8$$

$$\therefore x = 1$$

Question 8 — Worked Solution

(a) Find $g^{-1}(x)$

Let $y = (x+1)^2 - 4$. Rearrange:

$$(x+1)^2 = y + 4 \Rightarrow x + 1 = \sqrt{y+4}$$

(Positive root since $x \geq -1$, so $x+1 \geq 0$.)

$$x = \sqrt{y+4} - 1$$

$$\therefore g^{-1}(x) = \sqrt{x+4} - 1$$

(b) Domain and range of g^{-1}

Domain of g^{-1} = Range of g. At $x = -1$: $g(-1) = 0 - 4 = -4$ (minimum). So range of g is $y \geq -4$.

Domain of g^{-1}: $x \geq -4$

Range of g^{-1}: $y \geq -1$ (= domain of g)

(c) Sketch

g(x) is a parabola with vertex $(-1, -4)$, passing through $(0,-3)$ and $(1, 0)$ for $x \geq -1$. $g^{-1}(x)$ is its reflection in $y = x$. They intersect on $y = x$.

Question 9 — Worked Solution

Solve |3x – 2| = |x + 4|

Square both sides (valid since both sides are non-negative):

$$(3x-2)^2 = (x+4)^2$$

$$9x^2 - 12x + 4 = x^2 + 8x + 16$$

$$8x^2 - 20x - 12 = 0 \Rightarrow 2x^2 - 5x - 3 = 0$$

$$(2x+1)(x-3) = 0$$

$$\therefore x = 3 \quad \text{or} \quad x = -\frac{1}{2}$$

Question 10 — Worked Solution

Solve $|x^2 - 5| = 4$

Case 1: $x^2 - 5 = 4 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$

Case 2: $x^2 - 5 = -4 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

$$\therefore x = -3, \ -1, \ 1, \ 3$$

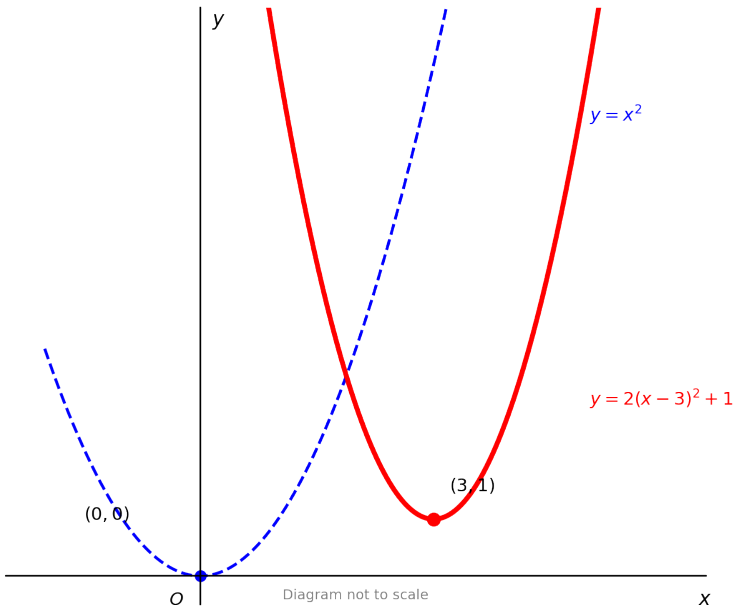
Question 11 — Worked Solution

(a) Transformations from y = x^2 to y = 2f(x – 3) + 1

Step 1: f(x – 3): translate the graph **3 units to the right** (parallel to x-axis). Vertex moves from (0,0) to (3,0).

Step 2: 2f(x – 3): **stretch by scale factor 2** in the y-direction (parallel to y-axis). Vertex stays at (3,0).

Step 3: 2f(x – 3) + 1: translate the graph **1 unit upward** (parallel to y-axis). Vertex moves to (3, 1).



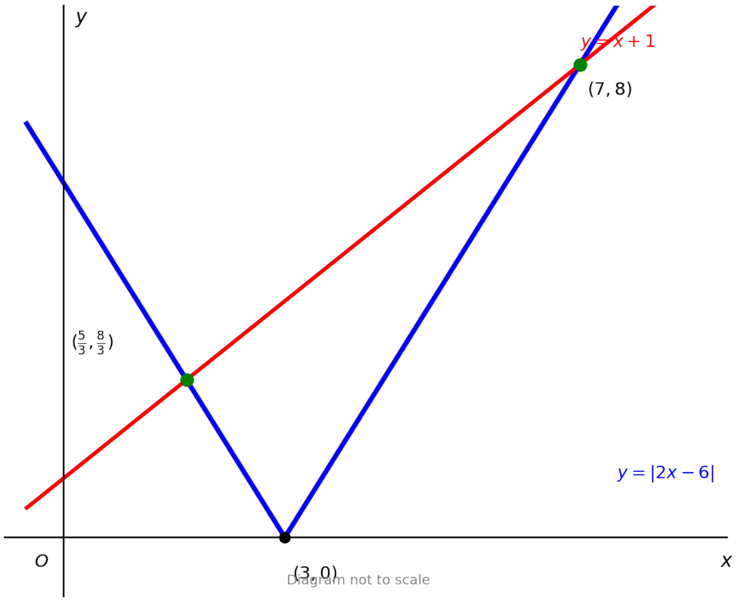
(b) Vertex

$$\therefore \text{Vertex} = (3,\ 1)$$

Question 12 — Worked Solution

(a) Sketch both graphs

$y = |2x – 6|$: V-shape with vertex at (3, 0). y-intercept at (0, 6). $y = x + 1$: straight line, y-intercept (0,1), x-intercept (–1, 0).



(b) Solve |2x – 6| = x + 1

Case 1: $2x – 6 = x + 1 \Rightarrow x = 7$. Check: $|14–6|=8=7+1$ ✓

Case 2: $2x – 6 = –(x+1) \Rightarrow 2x – 6 = –x – 1 \Rightarrow 3x = 5 \Rightarrow x = 5/3$. Check: $|10/3–6|=8/3=5/3+1$ ✓

$$\therefore x = \frac{5}{3} \quad \text{or} \quad x = 7$$